Networks, Barriers, and Trade

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Abstract

We study a non-parametric class of trade models with global production networks. We characterize their properties in terms of interpretable and measurable sufficient statistics. We provide both reduced-form results for measuring the sources of growth and structural results for conducting counterfactuals. Our main objectives are to show how commonly-used stylized models can give misleading results because of simplifying assumptions regarding intermediate inputs, factors, elasticities, and distortions, and to show how more complex models that do not rest on these assumptions work. As an example, accounting for nonlinear (non-Cobb-Douglas) production networks, with realistic complementarities in production, significantly raises the gains from trade relative to estimates in the literature. As another example, models with value-added production functions, no matter how well-calibrated, are incapable of simultaneously predicting the costs of tariff and non-tariff barriers to trade. Accounting for intermediates doubles the losses from tariffs, and this magnification is due to the double-marginalization of tariffs caused by global value chains and the fact that intermediates endogenously raise the elasticity of quantities with respect to tariffs. Better quantitative accuracy demands the use of more complicated, oftentimes computational, models. This paper seeks to help bridge the gap between computation and theory.

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1 Introduction

Trade economists increasingly recognize the importance of using large-scale computational general equilibrium models for studying trade policy questions. One of the major downsides of relying on purely computational methods is their opacity: computational models can be a black box, and it is sometimes hard to know which forces in the model drive specific results. On the other hand, it is increasingly recognized that relying on simple stylized models, while transparent and parsimonious, can result in unreliable quantitative predictions when compared to the large-scale models.

This paper seeks to provide a theoretical map of territory usually explored by machines. It studies output and welfare in open economies with disaggregated and interconnected production structures. We address two types of questions: (i) how to measure and decompose the sources of output and welfare changes, and (ii) how to predict the responses of output, welfare, as well as disaggregated prices and quantities, to changes in trade costs or tariffs. Our analysis is non-parametric and quite general, which helps us to isolate the common forces and sufficient statistics necessary to answer these questions without committing to a specific parametric set up.

We show how accounting for the details of the production structure (the input-output table and the elasticities of substitution) can theoretically and quantitatively change answers to a broad range of questions in open-economy settings. Simple stylized models, no matter how deftly calibrated, can get both the magnitude and even the direction of effects wrong.

In analyzing the structure of open-economy general equilibrium models, we emphasize their similarities and differences to the closed-economy models used to study growth and fluctuations. To fix ideas, consider the following fundamental theorem of closed economies. For a perfectly-competitive economy with a representative household and inelastically supplied factors

$$\frac{\mathrm{d}\log W}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_i} = \frac{sales_i}{GDP}, \tag{1}$$

where W is real income or welfare (measured as an equivalent variation), Y is real output or GDP, and A_i is a Hicks-neutral shock to some producer i.[1] Equation (1), also known as Hulten's Theorem, shows that the sales share of producer i is a sufficient statistic for understanding the impact of a shock on aggregate welfare, aggregate income, and aggre-

[1] Equation (1) is fundamental in the sense that it is a consequence of the first welfare theorem. Although versions of this result existed for a long time, at least since Domar (1961), the modern treatment is due to Hulten (1978).

gate output to a first order. Specifically, Hulten's theorem implies that, to a first order, any information beyond the sales share (for example, about the details of the production network, the returns to scale, or the elasticities of substitution) is macroeconomically irrelevant.

In this paper, we examine the extent to which the logic of (1) can be transported into international economics. We provide the open-economy analogues of equation (1), and show that although versions of Hulten's theorem continue to hold in open-economies, the sales-shares are no longer such universal sufficient statistics. Ultimately, there are two main barriers to blindly applying Hulten's theorem in an open-economy: first, in an open-economy, output and welfare are no longer the same since welfare depends on terms-of-trade but output does not (see e.g. Burstein and Cravino, 2015); second, much of trade policy concerns the effects of tariffs, which knocks out the foundation of marginal cost pricing and Pareto efficiency that Hulten's Theorem is built on. Our generalizations make clear precisely the conditions under which a naive-application of (1) to an open-economy is valid. Even when not directly applicable, it proves helpful to think in terms of (1), and deviations from it.

Notwithstanding the differences between open and closed economies, we also prove that, under some conditions, there exists a useful isomorphism between the two types of models. In particular, for any open-economy with nested-CES import demand there exists a companion (dual) closed economy, and the welfare effects of iceberg shocks in the open-economy are equal to the output effects of productivity shocks in the closed economy. This means that we can use results from the closed-economy literature, principally Hulten (1978) and Baqaee and Farhi (2017a), to characterize the effects of iceberg shocks on welfare up to the second-order. Our formulas provide a generalization of some of the influential insights of Arkolakis et al. (2012) to environments with disaggregated, non-loglinear input-output connections. Compared to the Cobb-Douglas loglinear production networks common in the literature (e.g. Costinot and Rodriguez-Clare, 2014; Caliendo and Parro, 2015), we find that accounting for nonlinear production networks significantly raises the gains from trade. Accounting for nonlinear input-output networks is as, or more important, as accounting for intermediates in the first place. For example, for the US, the gains from trade increase from 4.5% to 9% once we account for intermediates with a Cobb-Douglas network, but they increase further to 13% once we account for realistic complementarities in production. The numbers are even more dramatic for more open economies, for example, the gains from trade for Mexico go from 11% to 16% to 44.5%.

For most of the paper, we restrict ourselves to efficient economies, but extending the results to allow for arbitrary distorting wedges (e.g tariffs or markups) is straightforward.

To our knowledge, this is the first paper in the literature that derives comparative statics with respect to tariffs (in terms of model primitives) in a general production environment with intermediate goods. We show that, in general, the output losses to the world as a whole, and to the output of each country, from the imposition of tariffs or other distortions can be computed by an appropriate summing up of Harberger triangles, even in the absence of implausible compensating transfers. We provide explicit formulas for what these Harberger triangles are equal to in terms of microeconomic primitives. We explain how to adjust these formulas to obtain welfare losses. We show that the existence of global value chains dramatically increases the costs of protectionism by inflating both the area of each triangle *and* the weight used to aggregate the triangles. We show that simple (non-input-output) models, regardless of how they are calibrated, get either the area of the triangles or their weight wrong.

Intuitively, the weight on each triangle is just the sales share of the taxed good. Since input-output connections inflate sales relative to value-added, that means accounting for intermediates can inflate the weight each Harberger triangle receives. More subtly, the area of each triangle is also increased in the presence of intermediates. There are two reasons for this: first, global value chains mean that tariffs are compounded each time an unfinished good crosses the border, à la Yi (2003); second, in the presence of intermediates, the production of traded goods is more elastic with respect to tariffs, since, holding fixed the volume of trade, trade is a smaller portion of each individual agent's basket. Both of these effects combine, in roughly equal magnitudes, to amplify the cost of tariffs to the world economy. As an example, we find that a worldwide increase in import tariffs from zero to ten percent reduces world output by -0.43%. If we ignore input-output connections, this number is halved.

We include some empirical and quantitative applications of our results. We use our growth-accounting formulas to measure the size and direction of reallocations of resources across countries over time, and relate these to movements in the terms-of-trade for each country. Finally, as a proof-of-concept, we use a calibrated model to illustrate the quantitative importance of the forces we emphasize. We consider three suggestive exercises, each of which highlights a conceptual point: (1) reversing globalization, (2) Brexit, and (3) US tariffs on China. First, we consider the welfare changes associated with a reversal of globalization, achieved either via import tariffs or non-tariff trade barriers. We show that the latter is far more costly than the former, and link this to the fact that Harberger triangles are smaller than the loss trapezoids generated by non-trade barriers. We also show that for both tariffs and non-tariff barriers, accounting for the existence of global value chains is crucial. For some countries, the losses from protectionism are tripled or quadrupled once we account for the existence of input-output connections.

Second, we consider a simple Brexit scenario where the remaining members of the European Union place tariffs or non-tariff trade barriers on British imports. For this scenario, we focus on the distributional consequences of the shock across different primary factors. We find that the magnitude and even the sign of the distributional consequences depend on whether or not we account for input-output linkages. In particular, whereas low and medium skilled British workers gain from Brexit when we use value-added production functions, they lose once we account for the production network.

Last, we compute the effects of US tariffs on Chinese imports. We find that taking the network connections into account, magnifies the gains to Mexico by a factor of three, the gains to the US by a factor of five, and the losses to China by a factor of six. This last exercise emphasizes the interconnected nature of global trade, and the importance of considering the trading system as a whole when evaluating policy counterfactuals.

The outline of the paper is as follows. In Section 2, we set up the model and define the objects of interest. In Section 3, we derive some growth-accounting results useful for measurement. In Section 4, we establish the dual relationship between closed and open economies which can be used to generalize some of the results in Arkolakis et al. (2012) and Costinot and Rodriguez-Clare (2014). In Section 5, we derive comparative statics in terms of microeconomic primitives, useful for prediction. In Section 6, we extend our analysis to allow for distortions like tariffs, and we show that analytically, the costs of tariffs are very different to those of iceberg shocks. Our empirical and quantitative applications are in Section 7. All the proofs are in Appendix K.

Related Literature

At a high-level, this paper is related to the classic papers of Hulten (1978), Harberger (1964), and Jones (1965). We extend Hulten (1978) and prove growth-accounting formulas for open-economies; we extend Harberger (1964) and show that deadweight-loss triangles can be used to measure productivity and welfare losses from tariffs in general equilibrium, even in the absence of compensating transfers; we extend the hat-algebra of Jones (1965) beyond the $2 \times 2 \times 2$ cases with no input-output linkages that he considered, and use it to answer counterfactual questions.

More broadly, our paper is related to three literatures: the literature on the gains from trade, the literature on production networks, and the literature on growth accounting. We discuss each literature in turn starting witAcemoglu2015h the one on the gains (or losses) from trade. As far as we are aware, this is the first paper to characterize the comparative static response of income and output to changes in iceberg costs and tariffs non-parametrically in a model with a rich input-output structure. In particular, our results generalize some of the results in Arkolakis et al. (2012) and Costinot and Rodriguez-Clare (2014) to environments with non-linear input-output connections. Our framework generalizes the input-output models emphasized in Caliendo and Parro (2015), Caliendo et al. (2017), Morrow and Trefler (2017), Fally and Sayre (2018), and Bernard et al. (2019). Our results about the effects of trade in distorted economies also relates to Epifani and Gancia (2011), Arkolakis et al. (2015), Berthou et al. (2018), Bai et al. (2018). Our results also relate to work with non-parametric or semi-parametric models of trade like Adao et al. (2017) and Lind and Ramondo (2018) (though our analysis does not rely on the invertibility, or stability, of factor demand systems), as well as Allen et al. (2014), (although we do not impose a gravity equation). Finally, our characterization of how factor shares and prices respond to shocks is related to an incredibly deep literature, for example, Trefler and Zhu (2010), Elsby et al. (2013), Davis and Weinstein (2008), Feenstra and Sasahara (2017), Burstein and Vogel (2017), Artuç et al. (2010), Dix-Carneiro (2014), Galle et al. (2017), among others.

The literature on production networks has primarily been concerned with the propagation of shocks in closed economies, typically assuming a representative agent. For instance, Long and Plosser (1983), Acemoglu et al. (2012), Atalay (2017), Carvalho et al. (2016), Baqaee and Farhi (2017a,b), and Baqaee (2018), among many others. A recent focus of the literature, particularly in the context of open economies, has been to model the formation of links, for example Chaney (2014), Lim (2017), Tintelnot et al. (2018), and Kikkawa et al. (2018). Our approach, which builds on the results in Baqaee and Farhi (2017a,b), is different: rather than modelling the formation of links as a binary decision, we use a Walrasian environment where the presence and strength of links are endogenously determined by cost minimization and input-substitution subject to some production technology.

Finally, our growth accounting results are related to closed-economy results like Solow (1957), Hulten (1978), as well as to the literature extending growth-accounting to open economies, including Kehoe and Ruhl (2008) and Burstein and Cravino (2015). Perhaps closest to us are Diewert and Morrison (1985) and Kohli (2004) who introduce output indices which account for terms-of-trade changes. Our real income and welfare-accounting measures share their goal, though our decomposition into pure productivity changes and reallocation effects is different. In explicitly accounting for the existence of intermediate inputs, our approach also speaks to how one can circumvent the double-counting problem and spill-overs arising from differences in gross and value-added trade, issues studied by Johnson and Noguera (2012) and Koopman et al. (2014). Relative to these other papers, our approach has the added bonus of easily being able to handle inefficiencies and wedges.

2 Framework

In this section, we do the spadework of setting up the model and defining the key statistics of interest. We assume that there are no distortions. In Section 6, we extend our results to environments with distortions (e.g. markups, tariffs, taxes).[2]

2.1 Model

There is a set of countries C with representative households, a set of producers N producing different goods, and a set of factors F. Each producer and each factor is assigned to be within the borders of one of the countries in C. The sets of producers and factors inside country c are N_c and F_c. We assume a representative agent for each country in the main body of the paper in order to not clutter the exposition. In Appendix F, we precisely show how all our main results can easily be extended to cover situations in which there are heterogenous agents within each country.

Factors

In each country, the representative household c is endowed with some share Φ_{cf} of the supply L_f of each factor f. We take the quantities and ownership structure of factors as exogenously given.[3]

Households

The representative household in country c maximizes a homogenous-of-degree-one demand aggregator[4]

$$W_c = \mathcal{W}_c(\{c_{ci}\}_{i \in N}),$$

[2]Distortions can be represented as wedges (implicit or explicit taxes). Tariffs, markups, and financial frictions are wedges, but iceberg trade costs are not.

[3]In Appendix H, we discuss how to endogeneize factor supply by using a model à la Roy (1951) and discuss the connection of our results with those in Galle et al. (2017).

[4]In mapping our model to data, we interpret domestic "households" as any agent which consumes resources without producing resources to be used by other agents. Specifically, this means that we include domestic investment and government expenditures in our definition of "households" when we map this model to the data.

subject to the budget constraint

$$\sum_{i \in N} p_i c_{ci} = \sum_{f \in F} \Phi_{cf} w_f L_f + T_c,$$

where c_{ci} is the quantity of the good produced by producer i and consumed by household c, p_i is the price of good i, w_f is the wage of factor f, and T_c is an exogenous lump-sum transfer. The lump-sum transfer can be used to allow for trade imbalances like in Dekle et al. (2008).

Producers

Each producer i in country c produces a different good using a constant-returns-to-scale production function with the associated production function

$$y_i = A_i F_i \left(\{x_{ik}\}_{k \in N}, \{l_{if}\}_{f \in F_c} \right),$$

where y_i is the total quantity of good i produced, x_{ik} is intermediate inputs from k, l_{if} is factor inputs from f, and A_i is an exogenous Hicks-neutral productivity shifter. For convenience, we sometimes set $l_{if} = 0$ for $f \notin F_c$.

Generality

This set up is more general than it might appear at first glance. The assumption that production features constant returns to scale is without loss of generality. As pointed out by McKenzie (1959), neoclassical production functions are constant-returns-to-scale without loss of generality, since any decreasing-returns production function can always be written as a constant returns production function by adding quasi-fixed factors.[5]

The assumption that each producer produces only one output good is without loss of generality. One can always represent a multi-output production function as a single output production function by letting all but one of the outputs enter as negative inputs. Joint production is therefore allowed by the model.

The assumption that productivity shifters are Hicks-neutral is also without loss of generality. For example, an input-augmenting technical change for producer i's use of

[5] Increasing returns can also in principle be accommodated, but only to some limited extent, by allowing these quasi-fixed factors to be local "bads", i.e. to receive negative payments over some range. However, care must be taken because increasing returns introduce non-convexities in the cost minimization over variable inputs, and our formulas only apply when variable-input demand changes smoothly.

input k can be captured by introducing a fictitious producer buying from k and selling to i and hitting this fictitious producer with a Hicks-neutral productivity change.[6]

Finally, the assumption that there are no shocks to the composition of final demand is without loss of generality, since such shocks can be represented via relabeling as combinations of positive and negative productivity shocks.

Iceberg Trade Costs

We capture changes in iceberg trade costs as Hicks-neutral productivity changes to specialized importers or exporters (whose production functions represent the trading technology). The decision of where the trading technology should be located is ambiguous since it generates no value-added. We do not need to take a precise stand at this stage, but we note that this will matter for our conclusions regarding real country GDP changes (as pointed out by Burstein and Cravino, 2015). It is possible to place them in the exporting country, in the importing country, or in a new fictitious separate specialized country which contains no factors.

Equilibrium

Given productivities A_i and a vector of transfers satisfying $\sum_{c \in C} T_c = 0$, a general equilibrium is a set of prices p_i, intermediate input choices x_{ij}, factor input choices l_{if}, outputs y_i, and consumption choices c_{ci}, such that: (i) each producer chooses inputs to minimize costs taking prices as given; (ii) each household maximizes utility subject to its budget constraint taking prices as given; and, (iii) the markets for all goods and factors clear so that $y_i = \sum_{c \in C} c_{ci} + \sum_{j \in N} x_{ji}$ for all $i \in N$ and $L_f = \sum_{j \in N} l_{jf}$ for all $f \in F$.

2.2 Definitions

In this subsection, we define the statistics of interest. Although these definitions are standard to national income accountants, and the distinctions we stress may seem tedious, it turns out that they make all the difference for the economics of the model.

Nominal Output and Nominal Expenditure

Nominal output or Gross Domestic Product (GDP) for country c is the total final value of the goods produced in the country. It coincides with the total income earned by the

[6] The assumption that factors cannot be used in production across borders is also without loss of generality, since we can always introduce a new fictitious producer transforming a non-traded factor f in a country $c' \neq c$ into a traded intermediate input for producer i.

factors located in the country or nominal Gross Domestic Income (GDI):

$$GDP_c = \sum_{i \in N} p_i q_{ci} = \sum_{f \in F_c} w_f L_f,$$

where $q_{ci} = y_i 1_{\{i \in N_c\}} - \sum_{j \in N_c} x_{ji}$ is the net quantity of good $i \in N$ in the GDP of country c (which means that it can be positive or negative).

Nominal Gross National Expenditure (GNE) for country c is the total final expenditures of the residents of the country. In our model, it coincides with nominal Gross National Income (GNI) which is the total income earned by the factors owned by its residents and adjusted for international transfers:

$$GNE_c = \sum_{i \in N} p_i c_{ci} = \sum_{f \in F} \Phi_{cf} w_f L_f + T_c.$$

Nominal output or GDP and nominal GNE are *not* the same at the country level in general because they account for the value created by different sets of factors (and the different expenditures that they finance): the factors in the country vs. the factors owned by the residents of the country.

Of course, these differences vanish at the world level:

$$GDP = GNE = \sum_{f \in F} w_f L_f = \sum_{i \in N} p_i q_i = \sum_{i \in N} p_i c_i,$$

where $c_i = q_i = y_i$ with $c_i = \sum_{c \in C} c_{ci}$, $q_i = \sum_{c \in C} q_{ci}$, $T = 0$ with $T = \sum_{c \in C} T_c$.

We let world GDP be the numeraire, so that $GDP = GNE = 1$. All prices and transfers are expressed in units of this numeraire.

Real Output and Real Expenditure

We now define changes in real output and real expenditures at different levels of aggregation. We use Divisia indices throughout to separate quantity and price changes, and rely on their convenient aggregation properties.

The change in real output (real GDP) of country c and the corresponding deflator are

$$\mathrm{d}\log Y_c = \sum_{i \in N} \chi_i^{Y_c} \,\mathrm{d}\log q_{ci}, \quad \mathrm{d}\log P_{Y_c} = \sum_{i \in N} \chi_i^{Y_c} \,\mathrm{d}\log p_i,$$

where $\chi_i^{Y_c} = p_i q_{ci} / GDP_c$.[7,8]

[7]We slightly abuse notation since $q_{ci} \leq 0$ for $i \notin N_c$, in which case we define $\mathrm{d}\log q_{ci} = \mathrm{d}\, q_{ci} / q_{ci}$.

[8]Note that country real output is only defined in changes, and these changes cannot be integrated to

The change in real expenditure or welfare (real GNE) of country c and the corresponding deflator are

$$\mathrm{d}\log W_c = \sum_{i \in N} \chi_i^{W_c} \,\mathrm{d}\log c_{ci}, \quad \mathrm{d}\log P_{W_c} = \sum_{i \in N} \chi_i^{W_c} \,\mathrm{d}\log p_i, \tag{2}$$

where $\chi_i^{W_c} = p_i c_{ci}/GNE_c$. The fact that (2) measures the welfare of country c is a consequence of Shephard's lemma.

Changes in real output and in real expenditure are *not* the same at the country level in general. The difference comes from two sets of reasons. First, with border cross-border factor holdings and international transfers, changes in nominal output and in nominal expenditure are not the same in general. Second, changes in the price deflators for GDP and welfare are not the same in general.

Of course, these differences vanish at the world level so that $\mathrm{d}\log Y = \mathrm{d}\log W$ and $\mathrm{d}\log P_Y = \mathrm{d}\log P_W = \mathrm{d}\log P$, where

$$\mathrm{d}\log Y = \sum_{i \in N} \chi_i^{Y} \,\mathrm{d}\log q_i, \quad \mathrm{d}\log P_Y = \sum_{i \in N} \chi_i^{Y} \,\mathrm{d}\log p_i,$$

$$\mathrm{d}\log W = \sum_{i \in N} \chi_i^{W} \,\mathrm{d}\log c_i, \quad \mathrm{d}\log P_W = \sum_{i \in N} \chi_i^{W} \,\mathrm{d}\log p_i,$$

with $\chi_i^Y = p_i q_i/GDP$ and $\chi_i^W = p_i c_i/GNE$. Though, we must tread carefully since the change in real expenditure for the world, unlike the one for each country, is no longer a legitimate global measure of welfare, in the sense that it cannot be integrated to recover a social welfare function. However, there does exist a welfare function that, to a first order, coincides with changes in real world GNE/GDP. We discuss this issue in more detail in Section 6.[9]

It is important to note the following aggregation properties. Changes in country real output and real expenditures aggregate up to their world counterparts.[10]

recover a real GDP function. This means that the any discrete change in real output depends on the path of the change. The precise way to proceed is to index the economy by a continuous index (say time t), which indexes all the relevant shifters and all the equilibrium variables. We can then compute changes in real output between the initial period ($t = 0$) and some final period ($t = \tau$) as the integral of the infinitesimal real output changes along the resulting path. The differential change stated in the theorem is the real output change which obtains in the limit of small time intervals $\tau \to 0$: it is independent of the particular path of integration. The same goes at the world level for real output and real expenditure.

[9]Assume that there is a homothetic Bergson-Samuelson welfare function. Measure changes in world welfare following a shock by the proportional reduction in the post-shock consumptions of all goods by all consumers which would keep the level of world welfare unchanged. Then if there is no desire to redistribute at the initial point, changes in world welfare coincide with changes in world real expenditure up to the first order.

[10]Namely, $\mathrm{d}\log Y = \sum_{c \in C} \chi_c^Y \,\mathrm{d}\log Y_c$ and $\mathrm{d}\log W = \sum_{c \in C} \chi_c^W \,\mathrm{d}\log W_c$. This makes use of the following

Finally, the infinitesimal changes that we have defined for real output and real expenditure or welfare can be integrated or *chained* into discrete changes by updating the corresponding shares along the integration path. We denote the corresponding discrete changes by $\Delta \log Y$, $\Delta \log Y_c$, $\Delta \log W$, and $\Delta \log W_c$.

Input-Output Concepts

We define the Heterogenous-Agent Input-Output (HAIO) matrix to be the $(C+N+F) \times (C+N+F)$ matrix Ω whose ijth element is equal to i's expenditures on inputs from j as a share of its total revenues/income

$$\Omega_{ij} = \frac{p_j x_{ij}}{p_i y_i}.$$

The HAIO matrix Ω includes the factors of production and the households, where factors consume no resources (zero rows), while households produce no resources (zero columns). The Leontief inverse matrix is

$$\Psi = (I - \Omega)^{-1} = I + \Omega + \Omega^2 + \ldots.$$

The input-output matrix Ω records the *direct* exposures of one agent or producer to another, whereas the Leontief inverse matrix Ψ records instead the *direct and indirect* exposures through the production network.

It will sometimes be convenient to treat good and factors together and index them by $k \in N + F$ where we use the plus symbol to denote the union of these two sets. To this effect, we must slightly extend our definitions. We also write interchangeably y_k and p_k for L_k and w_k when $k \in F$. To capture the fact that the household endowment of the goods are zero, we define $\Phi_{ck} = 0$ for $(c, k) \in (C, N)$.

We define the *exposures* of real expenditure (welfare) and real output to each good and each factor. The exposures of country c's real expenditure (welfare) and real output to a good or factor k are

$$\lambda_k^{W_c} = \sum_{i \in N} \chi_i^{W_c} \Psi_{ik}, \quad \lambda_k^{Y_c} = \sum_{i \in N} \chi_i^{Y_c} \Psi_{ik},$$

where recall that $\chi_i^{W_c} = p_i c_{ci}/GNE_c$ and $\chi_i^{Y_c} = p_i q_{ci}/GDP_c$. The exposures of world real expenditure or welfare and real output to a good or factor k are

$$\lambda_k^W = \sum_{i \in N} \chi_i^W \Psi_{ik}, \quad \lambda_k^Y = \sum_{i \in N} \chi_i^Y \Psi_{ik},$$

definitions $\chi_c^Y = GDP_c/GDP$, $\chi_c^W = GNE_c/GNE$.

where recall that $\chi_i^W = p_i c_i / GNE$ and $\chi_i^Y = p_i q_i / GDP$.

Exposures of real output to good or factor k at the country and world levels have a direct connection to the sales of the producer:

$$\lambda_k^{Y_c} = 1_{\{k \in N_c + F_c\}} \frac{p_k y_k}{GDP_c}, \quad \lambda_k^Y = \frac{p_k y_k}{GDP}.$$

Hence, for example, λ_k^Y is just the sales share $\lambda_k = p_k y_k / GDP$ of k in world output or the *Domar weight*. Similarly $\lambda_k^{Y_c} = 1_{\{k \in N_c + F_c\}} (GDP/GDP_c) \lambda_k$ is the *local Domar weight* of k in country c.

We also define the following factor *income shares* as the shares in income or expenditure. The share of a factor f in the income or expenditure of country c and of the world are given by

$$\Lambda_f^c = \frac{\Phi_{cf} w_f L_f}{GNE_c}, \quad \Lambda_f = \frac{w_f L_f}{GNE},$$

where, from now on, we sometimes denote factors shares and exposures to factors or factor shares with a capital Λ to distinguish them from sales shares and exposures to non-factor producers λ.

In general, the exposures of welfare and real output to a good or factor k are *not* the same at the level of a country. Similarly, when applied to a factor f, these exposures are *not* the same as the income share of that factor at the level of a country. These differences disappear at the world level so that $\lambda_i^Y = \lambda_i^W = \lambda_i = p_i y_i / GDP$ for a good $i \in N$ and $\Lambda_f^Y = \Lambda_f^W = \Lambda_f = w_f L_f / GDP$ for a factor $f \in F$.

3 Comparative Statics: Ex-Post Sufficient Statistics

In this section, we characterize the response of real output and welfare to shocks (productivity, factor supply, and transfer shocks) at the country and world levels. Since iceberg trade costs can be represented as productivity shocks (to the trading technology), these characterizations extend to iceberg trade shocks. The results in this section are expressed in terms of expenditure shares, and (for welfare) changes in expenditure shares, essentially extending Hulten's theorem to open economies.

For changes in output, Hulten's theorem can be extended with little modification. However, for welfare, the extension is non-trivial and changes in welfare, in general, depend on changes in expenditure shares (specifically changes in factor shares). Since changes in factor shares are endogenous, this means that the results for welfare cannot be used for counterfactuals without additional information on how factor shares will respond. In Section 5, we provide the full characterization of how factor shares change in terms of microeconomic primitives, which allows us to use the welfare formulas in this section for counterfactuals.

Nevertheless, despite depending on endogenous objects, the welfare formulas in this section are useful for three reasons: (i) they provide intuition about why and how Hulten's theorem fails to describe welfare in open economies, (ii) they can be used to decompose changes in welfare into different sources *conditional* on the change in factor shares, and (iii) they can be combined with the results in Section 5 to perform counterfactuals.

3.1 Output-Accounting

We start with our output-accounting result: the response of real output (real GDP) to shocks. We state the result at the level of a country c and explain how to translate it to the level of the world.

Theorem 1 (Output-Accounting)**.** *The change in real output (real GDP) of country c to productivity shocks, factor supply shocks, and transfer shocks, is given by*

$$\mathrm{d}\log Y_c = \sum_{i\in N} \lambda_i^{Y_c}\,\mathrm{d}\log A_i + \sum_{f\in F} \Lambda_f^{Y_c}\,\mathrm{d}\log L_f.$$

where the exposures of real output to producers and factors are given by the local Domar weights $\lambda_i^{Y_c} = 1_{\{k\in N_c\}}(p_i y_i/GDP_c)$ and $\Lambda_f^{Y_c} = 1_{\{f\in F_c\}}(w_f L_f/GDP_c)$. The change $\mathrm{d}\log Y$ of world real output (GDP) can be obtained by simply suppressing the country index c.

Theorem 1 is an adaptation of Hulten's theorem to open economies. It implies that to a first order, a unit productivity shock to i moves real output in a country c by an amount equal to producer i's local Domar weight $\lambda_i^{Y_c}$. A counterintuitive implication of this equation is that to a first order, productivity shocks to foreign producers have *no* effect on domestic real output.[11] Since discrete changes in real output are obtained by chained integration of infinitesimal changes, the same counterintuitive implication actually holds globally. [12]

As emphasized by Burstein and Cravino (2015), productivity-accounting à la Hulten (1978) is the same in an open economy as it is in a closed economy. Country c's aggre-

[11] There are two important caveats to this statement: (i) shocks to foreign producers may change domestic local Domar weights, and thereby change the way local shocks affect the domestic economy (a nonlinear interaction), (ii) it is ceases to be true when the economy is no longer efficient. We shall discuss both of these issues at length in Sections 4 and 6.

[12] The same reasoning applies to factor supply shocks. Transfer shocks, no matter where they occur, have no effect on domestic real output to the first order and globally (holding fixed factor quantities).

gate productivity change can be measured by its Solow residual and is equal to the local Domar-weighted sum of productivity shocks of domestic producers:

$$\mathrm{d}\log Y_c - \sum_{f \in F} \Lambda_f^{Y_c} \,\mathrm{d}\log L_f = \sum_{i \in N} \lambda_i^{Y_c} \,\mathrm{d}\log A_i.$$

Since shocks to iceberg costs are just shocks to the productivities of trading technologies, iceberg trade shocks outside of the borders of country c have no effect on its real output or on its real productivity. In a small-open economy, with exogenous world prices, shocks to the terms of trade (the relative price of exports and imports) can also be modeled as shocks to the productivity of a trading technology. Folk wisdom and naive intuition then suggest that shocks to the terms of trade should have the same effect as negative domestic productivity shocks. Our result reinforces the observation by Kehoe and Ruhl (2008) that this intuition is invalid. Holding fixed factor quantities, real output (real GDP) and aggregate productivity only respond to shocks inside a country's borders.

3.2 Welfare-Accounting

Theorem 1 shows that a straightforward extension of Hulten's theorem holds in open economies for changes in real output. However, this is no longer true for changes in welfare (or real expenditure). We prove two expressions for changes in welfare, each providing an economically-interpretable decomposition of changes in welfare: the first emphasizes the role of terms-of-trade effects and the second the role of reallocation effects.

Terms of Trade Decomposition

We start with the more elementary terms-of-trade decomposition. A crucial point to observe is that changes in welfare, unlike those of output, depend on changes in factor shares.

Theorem 2 (Welfare-Accounting, Terms of Trade)**.** *The change in welfare of country c in re-*

sponse to productivity shocks, factor supply shocks, and transfer shocks can be decomposed into:[13]

$$\mathrm{d}\log W_c = \underbrace{\frac{\chi_c^Y}{\chi_c^W}\mathrm{d}\log Y_c}_{\Delta Output} + \underbrace{\frac{\chi_c^Y}{\chi_c^W}\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}}_{\Delta Terms\ of\ Trade} + \underbrace{\frac{1}{\chi_c^W}\mathrm{d}\,T_c + \sum_{f\in F}(\Lambda_f^c - \frac{\chi_c^Y}{\chi_c^W}\Lambda_f^{Y_c})\,\mathrm{d}\log\Lambda_f}_{\Delta Transfers\ and\ Net\ Factor\ Payments},$$

where the change in terms of trade $(\chi_c^Y/\chi_c^W)\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}$ *is*

$$\sum_{i\in N}(\lambda_i^{W_c} - \frac{\chi_c^Y}{\chi_c^W}\lambda_i^{Y_c})\,\mathrm{d}\log A_i + \sum_{f\in F}(\Lambda_f^{W_c} - \frac{\chi_c^Y}{\chi_c^W}\Lambda_f^{Y_c})\left(-\mathrm{d}\log\Lambda_f + \mathrm{d}\log L_f\right),$$

with $\chi_c^Y/\chi_c^W = GDP_c/GNE_c$*. The change* $\mathrm{d}\log W$ *of world real expenditure can be obtained by simply suppressing the country index* c*.*

To understand this result, consider for example a unit change in the productivity of producer i on the terms of trade. Intuitively, for given factor wages, the productivity shock affects the terms of trade of country c according to the difference between the country's exposures to producer i in real expenditure and in real output $\lambda_i^{W_c} - (\chi_c^Y/\chi_c^W)\lambda_i^{Y_c}$. The productivity also leads to endogenous changes in the wages of the different factors $\mathrm{d}\log w_f$, which given that factor supplies are fixed, coincide with the changes in their factor income shares $\mathrm{d}\log\Lambda_f$.[14] These changes in factor wages in turn affect the country's terms of trade according to the difference between the country's exposures to producer f in real expenditure and in real output $\Lambda_f^{W_c} - (\chi_c^Y/\chi_c^W)\Lambda_f^{Y_c}$.

At the country level, unlike real output, real expenditure or welfare *does* respond to productivity shocks outside the country in general because these shocks affect the terms of trade (and net factor payments). In particular, while shocks to iceberg trade costs outside a country do not affect its real output or its productivity, they do affect its real expenditure or welfare.

At the world level, there are no terms-of-trade effects (and no transfers or net factor payments). Furthermore, changes in real output and real expenditure or welfare and their corresponding deflators for each country aggregate up to their world counterparts. This

[13] When all factors inside a country are owned by the residents of that country, $\Lambda_f^c = \Lambda_f^{Y_c}$, and so net factor payments are zero. If in addition, there are no transfers so that $T_c = 0$, then $\chi_c^Y = \chi_c^W$ and our decomposition is invariant to changes in the numeraire. Outside of this case, the choice of numeraire influences the breakdown into changes in terms of trade and changes in transfers and net factor payments, but not the sum of the two.

[14] The formula actually still applies with endogenous factor supplies.

implies that changes in the country terms of trade sum up to zero:

$$\sum_{c\in C} \chi_c^W [(\chi_c^Y/\chi_c^W)\, \mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}] = \mathrm{d}\log P_Y - \mathrm{d}\log P_W = 0,$$

where $\chi_c^W = GNE_c/GNE$ and $\chi_c^Y = GDP_c/GDP$. Terms-of-trade effects can therefore be interpreted as zero-sum distributive effects. The same goes for transfers and net factor payments.

Reallocation Decomposition

We now present the reallocation decomposition. We follow Baqaee and Farhi (2017b) and define the $(C+N+F)\times(C+N+F)$ *allocation matrix* $\mathcal{X}$ as follows: $\mathcal{X}_{ij} = x_{ij}/y_j$ is the share of the quantity y_j of good j used by some agent i, where the indices i and j the households, factors, and producers. Every feasible allocation is defined by a feasible allocation matrix $\mathcal{X}$, a vector of productivities A, and a vector of factor supplies L. In particular, the equilibrium allocation gives rise to an allocation matrix $\mathcal{X}(A, L, T)$ which, together with A, and L, completely describes the equilibirum.[15]

Equilibrium welfare of country c can therefore be written as a function $\mathcal{W}_c(A, L, \mathcal{X})$ with $\mathcal{X} = \mathcal{X}(A, L, T)$. Differentiating yields a decomposition into two components: the direct or "pure" effect of changes in technology $\mathrm{d}\log A$ and $\mathrm{d}\log L$, holding the distribution of resources $\mathcal{X}$ constant; and the indirect effects arising from the equilibrium changes in the allocation of resources $\mathrm{d}\,\mathcal{X}$. In other words, this decomposition breaks down changes in welfare into: "pure" technology effects capturing changes in welfare from increased production of each good, holding fixed every agent's share of each good; and reallocation effects capturing changes in welfare from changes in the shares of each good consumed by each agent. The following theorem characterizes this decomposition.

Theorem 3 (Welfare-Accounting, Reallocation)**.** *The change in real expenditure or welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be decomposed into the "pure" effects of changes in technology and the effects of changes in the allocation of resources:*

$$\mathrm{d}\log W_c = \underbrace{\frac{\partial \log \mathcal{W}_c}{\partial \log L}\,\mathrm{d}\log L + \frac{\partial \log \mathcal{W}_c}{\partial \log A}\,\mathrm{d}\log A}_{\Delta Technology} + \underbrace{\frac{\partial \log \mathcal{W}_c}{\partial \mathcal{X}}\,\mathrm{d}\,\mathcal{X}}_{\Delta Reallocation},$$

[15]Since there may be multiplicity of equilibria, technically, the competitive equilibrium gives a correspondence from A to $\mathcal{X}$. In this case, we restrict attention to perturbations of isolated equilibria.

where the "pure" technology effects are given by

$$\frac{\partial \log \mathcal{W}_c}{\partial \log L} \,\mathrm{d} \log L + \frac{\partial \log \mathcal{W}_c}{\partial \log A} \,\mathrm{d} \log A = \sum_{f \in F} \Lambda_f^{W_c} \,\mathrm{d} \log L_f + \sum_{i \in N} \lambda_i^{W_c} \,\mathrm{d} \log A_i,$$

and the reallocation effects are given by

$$\frac{\partial \log \mathcal{W}_c}{\partial \mathcal{X}} \,\mathrm{d}\, \mathcal{X} = \sum_{f \in F} (\Lambda_f^c - \Lambda_f^{W_c}) \,\mathrm{d} \log \Lambda_f + (1/\chi_c^W) \,\mathrm{d}\, T_c.$$

The change $\mathrm{d} \log W$ *of world real expenditure can be obtained by simply suppressing the country index* c.[16,17]

To better understand this result, consider for example a unit change in the productivity of producer i. Intuitively, for given factor wages, the "pure" technology effect of the shock is given by the exposure $\lambda_i^{W_c}$ of the real expenditure of country c to this producer. The productivity shock also leads to endogenous changes in the wages of the different factors $\mathrm{d} \log w_f$, which, given that factor supplies are fixed, coincide with the changes in their factor income shares $\mathrm{d} \log \Lambda_f$.[18] The reallocation effect depends, for each factor f, on the change $\mathrm{d} \log \Lambda_f$ in the wage of that factor, and on the difference $\Lambda_f^c - \Lambda_f^{W_c}$ between the share of a that factor in the country's income and of the country's exposure of real expenditure to that factor.[19]

We can define the change in the *factoral terms of trade* to be $\sum_{f \in F} (\Lambda_f^c - \Lambda_f^{W_c}) \,\mathrm{d} \log w_f$. The previous discussion makes clear that with fixed factor supplies and in the absence of transfers, the reallocation effect is given by the change in the factoral terms of trade.[20] This is no longer true when factor supplies change since then changes in factor income shares $\mathrm{d} \log \Lambda_f$ which determine the reallocation effect $\sum_{f \in F} (\Lambda_f^c - \Lambda_f^{W_c}) \,\mathrm{d} \log \Lambda_f$ reflect both changes in the supplies of factors $\mathrm{d} \log L_f$ and changes in their wages $\mathrm{d} \log w_f$. The

[16] At the world level, and with a slight abuse of notation, the interpretation of the decomposition goes through provided we define the "pure" technology effects $(\mathrm{d} \log \mathcal{W} / \,\mathrm{d} \log L) \,\mathrm{d} \log L + (\mathrm{d} \log \mathcal{W} / \,\mathrm{d} \log A) \,\mathrm{d} \log A$ as changes in real expenditure at fixed prices holding the allocation matrix constant, and reallocation effects $(\mathrm{d} \log \mathcal{W} / \,\mathrm{d}\, \mathcal{X}) \,\mathrm{d}\, \mathcal{X}$ as the residual.

[17] We could also apply this decomposition to changes in real output: the "pure" technology effect would be given by Theorem 1, and the reallocation effect would be zero.

[18] The formula actually still applies with endogenous factor supplies.

[19] The same reasoning applies to shocks to factor supplies. Shocks to transfers contribute only through the reallocation effect directly by increasing income via the term $(1/\chi_c^W) \,\mathrm{d}\, T_c$ and also indirectly via the endogenous changes in the factor income shares $\mathrm{d} \log \Lambda_f$ that they bring about.

[20] Our definition can be seen as a formalization and a generalization of the "double factoral terms of trade" (in changes) discussed in Viner (1937). Our reallocation decomposition then provides a formal connection, missing in the analysis of Viner, between the changes in the factoral terms of trade and the change in welfare.

same remark applies when there are transfers.

Once again, we can see that at the country level, real expenditure or welfare, unlike real output, *does* respond to productivity shocks outside the country. Through the lens of this decomposition, it is because these shocks trigger reallocation effects.

Once we aggregate to the level of the world, of course, there are no reallocation effects.[21] Furthermore, the "pure" technology effect and the reallocation effect at the country level aggregate up to their world counterparts. This implies that the effects of country reallocations sum up to zero:

$$\sum_{c \in C} \chi_c^W (\mathrm{d} \log \mathcal{W}_c / \mathrm{d}\, \mathcal{X})\, \mathrm{d}\, \mathcal{X} = (\mathrm{d} \log \mathcal{W} / \mathrm{d}\, \mathcal{X})\, \mathrm{d}\, \mathcal{X} = 0.$$

Reallocation effects can therefore be interpreted as zero-sum distributive changes.

It is easy to imagine that if all production functions and all demand aggregators are Cobb-Douglas, then the allocation matrix is constant and so are factor income shares, implying that $\mathrm{d} \log \Lambda_f = 0$ for all factors f. This follows immediately from the results in Section 5. In this case, if in addition there are no shocks to transfers, then there are no reallocation effects and only "pure" technology effects. A Cobb-Douglas economy therefore provides a useful benchmark where there are only "pure" technology effects and no reallocation effects in changes in real expenditure or welfare.[22]

3.3 Terms of Trade vs. Reallocation

Theorems 2 and 3 provide two decompositions of changes in real expenditure or welfare with different economic interpretations: the terms-of-trade and reallocation decompositions. Both the reallocation effects and the terms-of-trade effects (and the net-factor-payments and transfer effects) can be interpreted as zero-sum distributive effects, and both of them can be written in terms of changes in factor shares. The goal of this section is to compare the two decompositions.

[21] This follows immediately from the since $\Lambda_f^Y = \Lambda_f^W = \Lambda_f$ for all factors f and since $\mathrm{d}\, T = 0$.

[22] Cole and Obstfeld (1991) also who showed that a Cobb-Douglas economy is a useful benchmark in another sense: in a Cobb-Douglas economy, in the absence of initial transfers, and with log utilities, there are no gains from allowing agents to trade financial assets to insure each other against country-specific productivity shocks. These observations are related. With log utilities, assuming that there are no initial transfers but not that the economy is Cobb Douglas, perfect risk sharing requires transfers $\mathrm{d}\, T_c = \chi_c^W = -\sum_{f \in F} \Lambda_f^c \,\mathrm{d} \log \Lambda_f$ ensuring that $\mathrm{d} \log W_f = -\mathrm{d} \log P_{W_f} = \sum_{i \in N} \lambda_i^{W_c} \,\mathrm{d} \log A_i + \sum_{f \in F} \Lambda_f^{W_c} \,\mathrm{d} \log w_f$ or equivalently $\mathrm{d} \log W_f = \sum_{i \in N} \lambda_i^{W_c} \,\mathrm{d} \log A_i + \sum_{f \in F} \Lambda_f^{W_c} \,\mathrm{d} \log L_f - \sum_{f \in F} \Lambda_f^{W_c} \,\mathrm{d} \log \Lambda_f$. These transfers become zero so that $\mathrm{d}\, T_c = 0$ when the economy is Cobb Douglas, and we then get $\mathrm{d} \log W_f = \sum_{i \in N} \lambda_i^{W_c} \,\mathrm{d} \log A_i + \sum_{f \in F} \Lambda_f^{W_c} \,\mathrm{d} \log L_f$.

Two Hulten-Like Results

To frame our discussion, it is useful to start by stating two different Hulten-like results for welfare in open economies. We call these "Hulten-like" results because they predict changes in welfare as a function of initial expenditure shares only *without* requiring information on changes in (endogenous) factor shares.

Corollary 1 (Welfare, Two Hulten-like Results). *In the following two special cases, Hulten-like results give changes in the welfare of a country c as exposure-weighted sums of productivity and factor supply shocks (and do not feature changes in factor shares).*

(i) Assume that country c receives no transfers from the rest of the world (balanced trade), there are no cross-border factor holdings, and international prices are exogenous and fixed (small-open economy). Then there are only real output effects, and no terms-of-trade, transfer effects, or net factor payment effects, so that the change in welfare is given by

$$\mathrm{d}\log W_c = \mathrm{d}\log Y_c = \sum_{f\in F_c} \Lambda_f^{Y_c}\,\mathrm{d}\log L_f + \sum_{i\in N_c} \lambda_i^{Y_c}\,\mathrm{d}\log A_i.$$

(ii) Assume either that the world economy is Cobb-Douglas or, if it is not, that we keep the allocation of resources (the allocation matrix) constant. Then there are only "pure" technology effects and no reallocation effects, so that the change welfare is given by:

$$\mathrm{d}\log W_c = \sum_{f\in F} \Lambda_f^{W_c}\,\mathrm{d}\log L_f + \sum_{i\in N} \lambda_i^{W_c}\,\mathrm{d}\log A_i.$$

Corollary 1 follows from Theorems 2 and 3. It shows that in some special cases, we can continue to use exposures to predict the effects of productivity and factor supply shocks on welfare in open economies.

The two Hulten-like results are very different. Focusing on productivity shocks, the elasticities $\mathrm{d}\log W_c / \mathrm{d}\log A_i$ of real expenditure to productivity shocks are given by exposures in real output $\lambda_i^{Y_c}$ in case (i) and by exposures in welfare $\lambda_i^{W_c}$ in case (ii).

The intuitions underlying the two Hulten-like results are also very different. The original Hulten theorem applies in a closed economy (e.g. the world) where there are neither terms-of-trade effects nor reallocation effects. In case (i), there are no terms-of-trade effects but there are reallocation effects. In case (ii), there are no reallocation effects, but there are terms-of-trade effects.

More generally, we can interpret the real output effects in Theorem 2 and the "pure" technology effects in Theorem 3 as Hulten-like terms, and the terms-of-trade effects (together with transfers and net factor payments) and reallocation effects as adjustment terms. As we saw earlier, these adjustment terms are zero-sum and depend on changes in factor shares.

Comparing the Terms-of-Trade and Reallocation Decompositions

Both decompositions can be applied at the level of a country and the world. Both decompositions isolate a distributive zero-sum term, which aggregates up to zero at the level of the world economy. These different distributive terms are responsible for departures from two different versions of Hulten's theorem.

The main difference between the two decomposition is their economic interpretations. We have already discussed these interpretations when we introduced the decompositions. We also provide some simple illustrative examples in Appendix I.

Beyond their differences in interpretation, the two decompositions have different robustness and aggregation properties, and different data requirements. In these regards, the reallocation decomposition has several advantages.

First the reallocation decomposition is based on general equilibrium counterfactual: "pure" changes in technology coincide with the change in real expenditure that would arise under the feasible counterfactual allocation which keeps the allocation of resources constant. This is not the case for the terms-of-trade decomposition: changes in real output are *not* the changes in real expenditure that would arise under a specified feasible counterfactual allocation.

Second, we will see in Section 5 that this particular general equilibrium counterfactual is extremely useful conceptually and intuitively in order to unpack our counterfactual results. This is because reallocation effects (but not "pure" technology effects) depend *only* on expenditure substitution by the different producers and households in the economy. By contrast, terms-of-trade effects also include technology effects.

Third, the reallocation decomposition is not sensitive to irrelevant changes in the environment, because it does not use changes in real output. This is not the case for the terms-of-trade decomposition: for example, assuming that changes in iceberg trade costs apply to the importers of a good or to its exporter simply produces different representations of the same underlying changes in the economy and is immaterial for changes in welfare, but it does modify the changes in terms of trade of the importers and of the exporter.

Outline of the Rest of the Paper

Theorems 2 and 3 show that changes in welfare, unlike changes in output, depend on changes in factor shares. In Section 5, we provide a full characterization of how factor shares change in terms of microeconomic primitives (ex-ante sufficient statistics).

Before doing so, in Section 4, we consider a simple case where the changes in factor shares can be deduced from changes in import shares. For such economies, we establish a duality result between open and closed economies, which allows us study gains from trade without solving for changes in world factor shares. It allows us to introduce a key concept, which we will use repeatedly in Sections 5 and Section 6: the input-output covariance operator. Section 4 therefore also serves as a good way to build intuition for the rest of the paper.

4 Duality Between Open and Closed Economies

In Section 3.1, we showed that shocks to a country's terms of trade do not act like domestic productivity shocks in the sense that they do not affect its real output or productivity. In this section, we show that such shocks do act like productivity shocks on the country's real expenditure or welfare. We do so by establishing a useful duality between the effects of foreign shocks to iceberg trade shocks in an open economy and the effects of domestic productivity shocks in a closed one. By bridging the open and closed economy literatures, this result allows us bring results from the (simpler) closed models to understand the behavior of open ones.

We prove that, under some conditions, the effects of iceberg trade shocks on real expenditure or welfare in an open economy can be analyzed by studying the effects of productivity shocks on real output in a companion closed economy. We can therefore shed light on the gains from trade in an open economy by leveraging the characterizations of the linear and nonlinear effects of productivity shocks on real output in closed economies provided respectively in Hulten (1978) and Baqaee and Farhi (2017a). These duality results build on a formula in Feenstra (1994) and can be seen as a generalization of some of results in ACR (Arkolakis et al., 2012). Unlike the other results in the paper, they do rely on the nested-CES parametric restriction.

4.1 Setup

We start by introducing our setup and defining some new input-output concepts.

Nested-CES Economies

Throughout this section, we restrict our attention to the class of models that belong to the nested-CES class, where each production function and each demand aggregator is a nested-CES function, with an arbitrary number of nests and arbitrary elasticities.

We adopt the following *standard form* representation. Since we restrict our attention to nested-CES models, we can relabel the network and rewrite the input-output matrix in such a way that: each producer corresponds to a single CES nest, with a single elasticity of substitution; the representative household in each country c consumes a single specialized good which, with some abuse of notation, we also denote by c. Importantly, note that this procedure, while it keeps the set of factors F unchanged, changes the set of producers, which, with some abuse of notation we still denote by N.

Input-Output Concepts

We use the following matrix notation throughout. For a matrix X, we define $X^{(i)}$ to be its ith row and $X_{(j)}$ to be its jth column. We define the *input-output covariance operator* to be

$$Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)}) = \sum_{l \in N+F} \Omega_{kl} \Psi_{li} \Psi_{lj} - \left(\sum_{l \in N+F} \Omega_{kl} \Psi_{li} \right) \left(\sum_{l \in N+F} \Omega_{kl} \Psi_{lj} \right).$$

It is the covariance between the ith and jth columns of the Leontief inverse using the kth row of the input-output matrix as the probability distribution. We make extensive use of the input-output covariance operator throughout the rest of the paper.

4.2 Duality Mapping

Consider an open economy c of the nested-CES form written in standard form. Each node of the network is then a producer $i \in N_c$ with a simple CES production function with a single elasticity of substitution θ_i with associated unit-cost function[23]

$$p_i = \frac{1}{A_i} \left(\sum_{j \in N+F_c} \Omega_{ij} p_j^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}}.$$

We construct a dual closed economy with the same set of producers $i \in N_c$ with CES production functions with the same set of elasticities θ_i and a HAIO matrix $\check{\Omega}$ given by

[23] Our results go through even when producers which do not directly use foreign inputs do not have CES production functions, but we assume for simplicity that they do.

$\check{\Omega}_{ij} = \Omega_{ij}/\Omega_{ic}$, where $\Omega_{ic} = \sum_{j \in N_c} \Omega_{ij}$ is the *domestic input share* of i. The unit-cost function of producer i in the dual closed economy is given by

$$\check{p}_i = \frac{1}{\check{A}_i} \left(\sum_{j \in N_c + F_c} \check{\Omega}_{ij} \check{p}_j^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}}.$$

We denote by $M_c \subseteq N_c$ the set of importing producers: the domestic producers which directly use foreign inputs in non-zero amounts. For such an importing producer $i \in M_c$, we sometimes use the notation $\epsilon_i = \theta_i - 1$ since this also corresponds to the trade elasticity of this producer.

4.3 Duality Results

To facilitate the exposition, we restrict ourselves to the case where the country c of interest has only one primary factor, which we call labor. We extend all the results to the case of multiple domestic factors in Appendix D.

We denote by W_c the welfare of country c, and by $\check{W}_c$ the welfare of the dual closed economy. Since the "inverted-hat" economy is closed, welfare is equal to real output $\check{W}_c = \check{Y}_c$.

Theorem 4 (Exact Duality)**.** *The discrete change in welfare $\Delta \log W_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to the discrete change in real output $\Delta \log \check{Y}_c$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i) \Delta \log \Omega_{ic}$.*

This duality allows us to leverage results from the literature on the real output effects of productivity shocks in closed-economy models to characterize the welfare effects of trade shocks in open economy models. We first provide an exact characterization result. We then use Hulten's theorem to characterize these effects to the first order, and Baqaee and Farhi (2017a) to characterize them to the second order.

It is useful to introduce the following mapping, which to every vector of price changes $\Delta \log \check{p}$ for the goods of the dual closed economy, associates a new vector of price changes $\Delta \log \check{p}' = T(\Delta \log \check{p}; \Delta \log \check{A})$ given by:

$$\Delta \log \check{p}_i' = -\Delta \log \check{A}_i + \frac{1}{1-\theta_i} \log \left(\sum_{j \in N_c + F_c} \check{\Omega}_{ij} e^{(1-\theta_i)\Delta \log \check{p}_j} \right).$$

It is easy to verify that $T(\cdot; \Delta \log \check{A})$ is a contraction mapping, the fixed-point of which gives the response of prices to productivity shocks in the dual closed economy: $\Delta \log \check{p} = \lim_{n\to\infty} T^n(\Delta \log \check{p}_{\text{init}}; \Delta \log \check{A})$, for all $\Delta \log \check{p}_{\text{init}}$. The response of welfare in the original economy is equal to the response of real output in the dual closed economy and is given by $\Delta \log W_c = \Delta \log \check{Y}_c = -\Delta \log \check{p}_c$, where recall that we denote by c the final good consumed by the representative agent of the dual closed economy.

This expression gives a representation of the exact response of output to productivity shocks via an infinite iteration of a nonlinear contraction mapping. Of course, when there is no reproducibility in the dual closed economy, then the fixed-point problem has a finite recursive structure moving from upstream producers to downstream producers, and leads to a closed-form expression.

Corollary 2 (First-Order Duality). *A first-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = \Delta \log \check{Y}_c \approx \sum_{i\in M_c} \check{\lambda}_i \Delta \log \check{A}_i,$$

where applying Hulten's theorem, $\check{\lambda}_i$ is the sales share or Domar weight of producer i in the dual closed economy.

Conditional on the size of the associated productivity shocks, the presence of intermediate inputs amplifies the effects of trade shocks much in the same way that the effect of intermediate inputs amplifies the effects of productivity shocks in closed economies. This is because (gross) sales shares are greater than (net) value-added shares, reflecting and intermediate-input multiplier discussed by, among others, Jones (2011). This observation is behind the findings of Costinot and Rodriguez-Clare (2014) that allowing for intermediate inputs significantly increases gains from trade.

Corollary 3 (Second-Order Duality). *A second-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = \Delta \log \check{Y}_c \approx \sum_{i\in M_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2}\sum_{i,j\in M_c} \frac{d^2 \log \check{Y}_c}{\mathrm{d} \log \check{A}_j \,\mathrm{d} \log A_i} \Delta \log \check{A}_j \Delta \log \check{A}_i,$$

where applying Baqaee and Farhi (2017a),

$$\frac{d^2 \log \check{Y}_c}{\mathrm{d} \log \check{A}_j \,\mathrm{d} \log \check{A}_i} = \frac{\mathrm{d}\, \check{\lambda}_i}{\mathrm{d} \log \check{A}_j} = \sum_{k\in N_c} (\theta_k - 1)\check{\lambda}_k Cov_{\check{\Omega}^{(k)}}\left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)}\right).$$

We can re-express the change in welfare in the original open economy as

$$\Delta \log W_c = \Delta \log \check{Y}_c \approx \sum_{i \in M_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2} \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right).$$

We start by discussing the second equation. It follows from Hulten's theorem that $\mathrm{d} \log \check{Y}_c / \mathrm{d} \log \check{A}_i = \check{\lambda}_i$. This immediately implies that $\mathrm{d}^2 \log \check{Y}_c / (\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i) = \mathrm{d} \check{\lambda}_i / \mathrm{d} \log \check{A}_j$. The term $(\theta_k - 1)\check{\lambda}_k Cov_{\check{\Omega}^{(k)}}(\check{\Psi}_{(i)}, \check{\Psi}_{(j)})$ captures the direct and indirect increase in expenditure on i in response to a shock to j because of substitution by k across its inputs. The term $\check{\lambda}_k$ is the total sales of k. The term $\theta_k - 1$ determines how much k substitutes expenditure towards ($\theta_k > 1$) or away from ($\theta_k < 1$) inputs which get relatively cheaper. The vector $\Psi_{(j)}$ captures the change in the input-price vector in response to the shock to j. The vector $\Psi_{(i)}$ captures how much an increase in expenditure on each input increases expenditure on i. These effects must be summed over producers k to determine the change $\mathrm{d} \check{\lambda}_i / \mathrm{d} \log \check{A}_j$ in the sales share of i in response to a shock to j.

The third equation in the corollary indicates that the ultimate impact of the shock depends on how heterogeneously exposed each producer k is to the average productivity shock via its different inputs as captured by the term $Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right)$, and on whether these different inputs are complements ($\theta_k < 1$), substitutes $(\theta_k > 1)$ or Cobb-Douglas $\theta_k = 1$. It indicates that complementarities lead to negative second-order terms which amplify negative shocks and mitigate positive shocks. Conversely, substitutabilities lead to positive second-order terms which amplify negative shocks and mitigate positive shocks. Finally, there are no second-order terms in the Cobb-Douglas case.

Duality with and Industry Structure

To discuss these results further, it is useful to assume that there is an *industry structure*: producers are grouped into industries and the goods produced in any given industry are aggregated with a CES production function; and all other agents only use aggregated industry goods. In this case, all domestic producers in a given industry are uniformly exposed to any other given domestic producer. This implies that in Corollary 3, only the elasticities of substitution across industries receive non-zero weights. The elasticities of substitution across producers within industries receive a zero weight, and they only matter via their influence on the productivity shocks through the trade elasticities.

In fact, the matrix $\check{\Omega}$ of the dual closed economy can be specified entirely at the industry level where the different producers are the different industries $\iota \in \mathcal{N}_c$. Given the productivity shocks $\Delta \log \check{A}_\iota$ to the importing industries $\iota \in \mathcal{M}_c$, Theorem 4 and Corollaries 2 and 3 can then be applied at the industry level, with this industry level input-output matrix, and with only elasticities of substitution across industries.

Many cases considered in the literature have such an industry structure, and impose the additional assumption that all the elasticities of substitution across industries (and with the factor) in production and in consumption are unitary (but those within industries are above unity). This makes the dual closed economy Cobb-Douglas. Such assumptions are made for example by ACR, Costinot and Rodriguez-Clare (2014), and Caliendo and Parro (2015). In this Cobb-Douglas case, the dual closed economy is exactly log-linear in the dual productivity shocks $\Delta \log \check{A}_\iota$. The effects of shocks to iceberg trade costs or to productivities outside of the country then coincide with the first-order effects of the dual shocks given by Corollary 2. Their second-order effects given by Corollary 3 are zero, and the same goes for their higher-order effects.

For example, we can recover the basic result of ACR by assuming that there is a single industry ι producing only from labor so that $\check{\lambda}_\iota = 1$. In this case, we get $\Delta \log W_c = \Delta \log \check{A}_\iota$ as an exact expression. We can also recover the extension of the ACR result by Costinot and Rodriguez-Clare (2014) to allow for multiple industries and input-output linkages, but restricting elasticities of substitution across industries to be unitary. In this case, we get $\Delta \log W_c = \sum_{\iota \in \mathcal{M}_c} \check{\lambda}_\iota \Delta \log \check{A}_\iota$ as an exact expression.

Our results therefore generalize some of the insights of ACR and of Costinot and Rodriguez-Clare (2014) to models with input-output linkages and where elasticities of substitution across industries (and with the factor) are not unitary. In such models, the dual closed economy is no longer Cobb-Douglas. Deviations from Cobb Douglas generate nonlinearities, which can either mitigate or amplify the effects of the shocks depending on whether there are complementarities or substituabilities, and with an intensity which depends on how heterogeneously exposed the different producers are to the shocks.

Corollary 4 (Exact Duality and Nonlinearities with an Industry Structure)**.** *For country c with an industry structure, we have the following exact characterization of the nonlinearities in welfare changes of the original open economy.*

(i) (Industry Elasticities) Consider two economies with the same initial input-output matrix and industry structure, the same trade elasticities and changes in domestic input shares, but with lower elasticities across industries for one than for the other so that $\theta_\kappa \leq \theta'_\kappa$ *for all industries* κ. *Then* $\Delta \log W_c = \Delta \log \check{Y}_c \leq \Delta \log W'_c = \Delta \log \check{Y}'_c$ *so that negative (positive) shocks have larger negative (smaller positive) welfare effects in the economy with the lower industry elasticities.*

(ii) (Cobb-Douglas) Suppose that all the elasticities of substitution across industries (and with

the factor) are equal to unity ($\theta_\kappa = 1$), then $\Delta \log W_c = \Delta \log \check{Y}_c$ is linear in $\Delta \log \check{A}$.

(iii) *(Complementarities) Suppose that all the elasticities of substitution across industries (and with the factor) are below unity ($\theta_\kappa \leq 1$), then $\Delta \log W_c = \Delta \log \check{Y}_c$ is concave in $\Delta \log \check{A}$, and so nonlinearities amplify negative shocks and mitigate positive shocks.*

(iv) *(Substituabilities) Suppose that all the elasticities of substitution across industries (and with the factor) are above unity ($\theta_\kappa \geq 1$), then $\Delta \log W_c = \Delta \log \check{Y}_c$ is convex in $\Delta \log \check{A}$, and so nonlinearities mitigate negative shocks and amplify positive shocks.*

(v) *(Exposure Heterogeneities) Suppose that industry κ is uniformly exposed to the shocks as they unfold, so that $Var_{\check{\Omega}_s^{(\kappa)}}\left(\sum_{\iota \in \mathcal{M}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota\right) = 0$ for all s where s indexes the dual closed economy with productivity shocks $\Delta \log \check{A}_{\iota,s} = s \Delta \log \check{A}_\iota$, then $\Delta \log W_c = \Delta \log \check{Y}_c$ is independent of θ_κ. Furthermore*[24]

$$\begin{aligned}\Delta \log W_c = \Delta \log \check{Y}_c = &\sum_{\iota \in \mathcal{M}_c} \check{\lambda}_\iota \Delta \log \check{A}_\iota \\ &+ \int_0^1 \sum_{\kappa \in \mathcal{N}_c} (\theta_\kappa - 1) \check{\lambda}_{\kappa,s} Var_{\check{\Omega}_s^{(\kappa)}}\left(\sum_{\iota \in \mathcal{M}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota\right)(1-s)ds.\end{aligned}$$

These results (ii), (iii), and (iv) follow immediately from Corollary 3 applied at the industry level, which allows us to determine that at every point, the Hessian of the function $\Delta \log \check{Y}_c(\Delta \log \check{A})$ is null in case (ii), negative semi-definite in case (iii), and positive semi-definite in case (iv). The same logic can be used to prove a local version of (i) since the Hessians of the two economies at the original point are ordered (using the semi-definite condition partial ordering). Similar arguments can be used to derive a local version of (v).

These results can also be derived using $\Delta \log W_c = \Delta \log \check{Y}_c = -\Delta \log \check{p}_c$ with $\Delta \log \check{p} = \lim_{n \to \infty} T^n(\Delta \log \check{p}_{\text{init}}; \Delta \log \check{A})$, where the mapping $\Delta \log \check{p}' = T(\Delta \log \check{p}; \Delta \log \check{A})$ is applied at the industry level. Indeed, the mapping is always linear in $\Delta \log \check{A}$. If all the elasticities of substitution across industries are below (above) unity, then the mapping is convex (concave) in $\Delta \log p$, and hence the mapping preserves convexity (concavity) in $\Delta \log A$. This immediately implies (ii), (iii), and (iv). The result (i) also follows from the fact that the LE and HE mappings are increasing and monotonically ordered.

[24] In the Cobb-Douglas case, the structure of the domestic network of the dual closed economy is irrelevant since the sales shares are sufficient statistics. Outside of this case, the structure of the network matters in general beyond the first order. There is one non-Cobb-Douglas case where it does not and where we get a network-irrelevant closed-form solution: when all the elasticities across industries are uniform ($\theta_\kappa = \theta$) and when the domestic upstream supply chains of the different importing industries are disjointed set, we get $\Delta \log W_c = \Delta \log \check{Y}_c = \left(\sum_{\iota \in \mathcal{M}_c} \check{\lambda}_\iota e^{(\theta-1)\Delta \log \check{A}_\iota}\right)/(\theta - 1)$.

The formula in (v) can be obtained by integrating by parts the path-integral version of Hulten's theorem $\Delta \log W_c = \Delta \log \check{Y}_c = \int_0^1 \check{\lambda}_{\iota,s} \, \mathrm{d} \log \check{A}_{\iota,s}$ using Corollary 3 to get an expression for $\mathrm{d} \log \check{\lambda}_{\iota,s}$. The formula immediately implies the associated result. Unlike the other results, it depends on an assumption on endogenous equilibrium objects, namely $Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) = 0$ for $s \in (0,1)$. This assumptions sometimes follows immediately from the structure of the network at the original point (this will be the case in the critical foreign input example below).

Example: Critical Foreign Inputs

We provide a simple example of a country c depicted in Figure 1. The only traded good is energy E.[25] The representative household in the country consumes domestic goods 1 through to N with some elasticity of substitution σ, with equal sales shares $1/N$ at the initial point. Some fraction of goods, goods 1 through to M, are made via labor L and a composite energy good E with an elasticity of substitution θ, with an initial energy share $(N/M)\check{\lambda}_E$. The composite energy good is a CES aggregate of domestic and foreign energy with elasticity of substitution $\theta_E > 1$. Domestic energy E, as well as the rest of the consumption goods, goods $M+1$ through to N, are made using only domestic labor. We assume that the elasticity of substitution in production $\theta < 1$ and that production has stronger complementarities than consumption $\theta < \sigma$.

Consider an increase in iceberg trade costs which increases the cost of import of foreign energy. The welfare effect of this trade shock is the same as that of a negative productivity shock

$$\Delta \log \check{A}_E = -\frac{1}{\varepsilon_E} \Delta \log \check{\Omega}_{Ec} < 0,$$

to the energy producer of the dual closed economy, where $\varepsilon_E = \theta_E - 1$ is the trade elasticity of the energy composite good E and $\Delta \log \Omega_{Ec}$ is the change of its domestic expenditure share.

The exact expression for the impact of the trade shock on welfare can be found in closed form by exploiting the recursive structure of the contraction mapping because this example features no reproducibility:

$$\Delta \log W_c = -\frac{1}{1-\sigma} \log \left(\frac{M}{N} (\frac{N}{M} \check{\lambda}_E e^{-(1-\theta) \Delta \log \check{A}_E} + 1 - \frac{N}{M} \check{\lambda}_E)^{\frac{1-\sigma}{1-\theta}} + \frac{N-M}{N} \right).$$

[25]This example is an open-economy version of an example in Baqaee and Farhi (2017a).

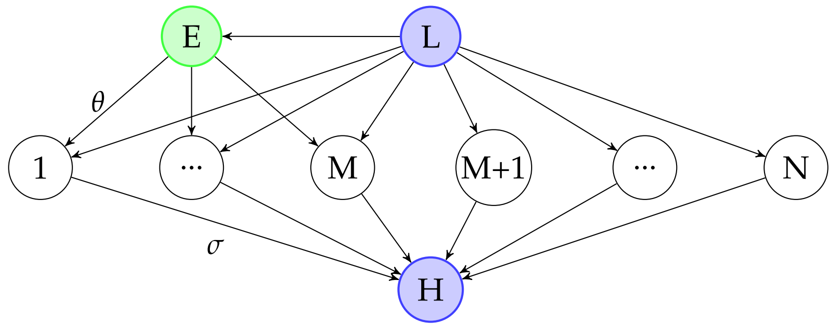


Figure 1: An illustration of the economy with a key input E. Each industry has different shares of labor and energy and substitutes across labor and energy with elasticity $\theta < 1$. The household can substitute across goods with elasticity of substitution $\sigma > \theta$. Energy is a traded good, which can either be produced domestically or sourced from the rest of the world, with an elasticity of substitution $\theta_E > 1$ between the two.

The second-order expression given by Corollary 3 is more transparent:

$$\Delta \log W_c \approx \check{\lambda}_E \Delta \log \check{A}_E + \frac{1}{2}\check{\lambda}_E \left((\sigma - 1)\check{\lambda}_E(\frac{N}{M} - 1) + (\theta - 1)(1 - \frac{N}{M}\check{\lambda}_E) \right) (\Delta \log \check{A}_E)^2.$$

When $M = N$, energy becomes a universal input, and the elasticity of substitution in consumption σ drops out of the expression. This is because the different goods 1 through N are uniformly exposed to the trade shock, and so substitution by the household is irrelevant. Since $\theta < 1$, nonlinearities captured by the second-order term amplify the negative welfare effects of the trade shock. This is because complementarities between energy and labor imply that the sales share of energy $\check{\lambda}_E$ increases with the shock, thereby amplifying its negative effect.

When $M < N$, the elasticity of substitution in consumption σ matters. Since $\sigma > \theta$, the nonlinear adverse effect of the trade shock is reduced compared to the case $M = N$ when we keep the initial sales share of energy $\check{\lambda}_E$ constant. This is true generally but the effect is easiest to see when $\sigma > 1$ since the household can now substitute away from energy-intensive goods, which mitigates the increase of the sales share of energy $\check{\lambda}_E$, and hence the negative welfare effects of the shock. These effects are stronger, the lower is M, i.e. the more heterogeneous are the exposures of the different goods to energy.

These effects are absent in the cases analyzed by ACR and Costinot and Rodriguez-Clare (2014) who make the Cobb-Douglas assumption $\sigma = \theta = 1$, which renders the model log-linear and eliminates all nonlinearities.

Quantitative Gains from Trade: Intermediate Inputs and Nonlinearities

We apply our duality results using the World Input-Output Database (WIOD) (see Timmer et al., 2015) to study the gains from trade by comparing the welfare losses from moving different countries to autarky. The WIOD contains the expenditures of each industry in each country on intermediate input purchases from every other industry in every other country. It also contains data on final consumption demand. We use data from 2008, which is the final year in the 2013 release of the data. The dataset has 41 countries, one of which is an aggregate Rest-of-World country, and each country has 30 industries. For more details on the industry classification scheme and the treatment of the data, see Appendix A.

We assume that production takes a nested CES form, where σ is the elasticity of substitution across industries in consumption, ζ is the elasticity of substitution between value-added and intermediate inputs, θ is the elasticity of substitution across industries in intermediate input use, ε_i is the elasticity of substitution between domestic and foreign varieties in industry i. The dual productivity shocks to the importing producers corresponding to a move to autarky of the original open economy are given by $\Delta \log \check{A}_i = -(1/\varepsilon_i) \log \Omega_{ic}$.

To calibrate the elasticities of substitution, we use the estimates from Caliendo and Parro (2015) for the trade elasticities ε_i. Our benchmark calibration is the far right column,where we set the elasticity of substitution across industries $\theta = 0.2$, the one between value-added and intermediates $\zeta = 0.5$, and the one in consumption $\sigma = 0.9$. These elasticities are broadly consistent with the estimates of Atalay (2017), Boehm et al. (2015), Herrendorf et al. (2013), and Oberfield and Raval (2014). Overall, the evidence suggests that these elasticities are all less than one (sometimes significantly so).

The results of this exercise are in Table 1 for different values of the elasticities of substitution (σ, ζ, θ). The first column replicates the results of a multi-sector value-added model without intermediate inputs and with the Cobb-Douglas assumption $(\sigma, \zeta, \theta) = (1, 1, 1)$, reported in Costinot and Rodriguez-Clare (2014).[26] The second column replicates the results of an a model which allows for intermediate inputs but maintains the Cobb-Douglas assumption, also reported in Costinot and Rodriguez-Clare (2014). As expected, allowing for intermediate inputs increases gains from trade. This is because of the first-order or log-linear effect captured by Corollary 2: it reflects the fact that abstracting away from intermediate inputs reduces the volume of imports relative to GDP. The other columns continue to allow for intermediate inputs, but deviates from the Cobb-Douglas assumption,

[26]Since the value-added version of the model has no intermediate inputs, the production elasticities θ and ζ are irrelevant.

(σ, ζ, θ)	VA $(1, 1, 1)$	$(1, 1, 1)$	$(1, 0.5, 0.6)$	$(0.9, 0.5, 0.2)$
FRA	9.8%	18.5%	24.7%	30.2%
JPN	2.4%	5.2%	5.5%	5.7%
MEX	11.5%	16.2%	21.3%	44.5%
USA	4.5%	9.1%	10.3%	13.0%

Table 1: Gains from trade for a selection of countries. The first column is a multi-sector value-added economy with no intermediate inputs and with the Cobb-Douglas assumption. The second column allows for intermediate inputs but maintains the Cobb-Douglas assumption in the direction of complementarities. The other columns allow for intermediate inputs and relax the Cobb-Douglas assumption. The microeconomic trade elasticities are the same across all columns and taken from Caliendo and Parro (2015), so the size of the trade shock to each industry is the same across all columns.

giving rise to nonlinearities. Moving across columns towards more complementarities increases the gains from trade. This is because of the nonlinear effect captured by Corollary 3: more complementarities magnify gains from trade by increasing nonlinearities.

The magnitudes of these different effects are different across countries. The importance of accounting for intermediate inputs is largely independent of the degree of openness of the country. By contrast, the importance of accounting for nonlinearities does depend on the degree of openness: the more open the country, the larger are the dual productivity shocks, and the more nonlinearities matter. Overall, it seems that nonlinearities are as important as intermediate goods to the study of gains from trade.

5 Comparative Statics: Ex-Ante Sufficient Statistics

In Section 3, we related responses to shocks of real output and welfare to exposures and changes in factor income shares (ex-post sufficient statistics). In Section 4 we studied a special case where changes in welfare can be predicted without solving directly for changes in factor shares. In this section we characterize the responses of factor income shares to shocks as a function of sufficient-statistic microeconomic primitives: the HAIO matrix and elasticities of substitution in production and in consumption (ex-ante sufficient statistics). The results of this section can then be combined with Theorem 3 to conduct counterfactuals. We also characterize the responses to shocks of all prices and quantities. We focus on productivity shocks because shocks to factor supplies and to iceberg costs are special cases of productivity shocks. Transfer shocks are covered in Appendix G.

Throughout this section, we restrict our attention to the class of models that belong to the nested-CES class written in standard-form. The reason for this is clarity not tractability. We refer the reader to Appendix C for a discussion of how to generalize all of our results and intuitions to arbitrary economies with non-nested-CES production functions and demand aggreators.

5.1 Comparative Statics

We define two matrices. The first is the $(N+F)\times(N+F)$ "propagation-via-substitution" matrix Γ whose ijth element is

$$\Gamma_{ij} = \sum_{k\in N} (\theta_k - 1)\frac{\lambda_k}{\lambda_i} Cov_{\Omega^{(k)}}\left(\Psi_{(i)}, \Psi_{(j)}\right),$$

and which encodes the sort of substitutions by all producers discussed in Section 4. The second is the $(N+F)\times F$ "propagation-via-redistribution" matrix Ξ whose ifth element is

$$\Xi_{if} = \sum_{c\in C} \frac{\lambda_i^{W_c} - \lambda_i}{\lambda_i} \Phi_{cf}\Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor, and which encodes the redistribution of income across the different households in the different countries and its effects given their different expenditure patterns.

Factor Shares and Sales Shares/Domar Weights

We start with a characterization of the responses of factor income shares.

Theorem 5 (Factor Shares and Sales Shares/Domar Weights)**.** *The changes in the factor income shares (factor Domar weights) in response to a productivity shock to producer i are the solution of the following system of linear equations:*

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i} = \Gamma_{fi} - \sum_{g\in F}\Gamma_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i} + \sum_{g\in F}\Xi_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i}.$$

Given these changes in factor income shares, the changes in producer sales shares (producer Domar weights) in response to a productivity shock to producer i are:

$$\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log A_i} = \Gamma_{ji} - \sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i} + \sum_{g\in F}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i}.$$

More generally, we can use the characterization of the responses of factor shares to characterize the responses of the input-output matrix, of the Leontief inverse matrix, and of all the income shares and all the exposures in real output and real expenditure or welfare at the country and world levels.[27]

Consider for example the response $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_i$ of the income share of factor f to a positive unit shock to the productivity of producer i. For fixed factor prices, every producer k will substitute across its inputs in response to this shock. Suppose that $\theta_k > 1$, so that producer k substitutes (in expenditure shares) *towards* those inputs j that are more reliant on producer i, captured by Ψ_{ji}, the more so, the higher is $\theta_k - 1$. Now, if those inputs are also more reliant on factor f, captured by a high $Cov_{\Omega^{(k)}}\left(\Psi_{(f)}, \Psi_{(i)}\right)$, then substitution by k will increase demand for factor f and hence the income share of factor f. These substitutions, which happen at the level of each producer k, must be summed across producers leading to the first propagation-via-substitution term Γ_{fi}.

This shock also also triggers changes in factor prices which then sets off additional rounds of substitution in the economy that we must account for. The change in the price of each factor g is given by $\mathrm{d}\log w_g/\,\mathrm{d}\log A_i = \mathrm{d}\log\Lambda_g/\,\mathrm{d}\log A_i$. The effect on the share of factor f is the same as that of a set of equivalent negative productivity shock to the different factors, leading to the second propagation-via-substitution term $-\sum_{g\in F}\Gamma_{fg}\,\mathrm{d}\log\Lambda_g/\,\mathrm{d}\log A_i$.

These changes in factor prices also change the distribution of income across households in different countries. This in turn affects the demand for the factor f since the different households are differently exposed, directly and indirectly, to factor f, leading to the propagation-via-redistribution term $\sum_{g\in F}\Xi_{fg}\,\mathrm{d}\log\Lambda_g/\,\mathrm{d}\log A_i$.

These formulas show that Cobb-Douglas assumptions, prevalent in the literature which incorporates production networks in trade models for their analytical convenience, are also special (see e.g. Costinot and Rodriguez-Clare, 2014; Caliendo and Parro, 2015). Whenever $\theta_k = 1$, the term accounting for expenditure substitution by producer k in the propagation-via-substitution matrix Γ is equal to zero. We will return to this issue in Section 4 and show that taking into account the fact that many empirical estimates of

[27] For example, we have:

$$\frac{\mathrm{d}\log\Omega_{lm}}{\mathrm{d}\log A_i} = (\theta_l - 1)\frac{1}{\Omega_{lm}}Cov_{\Omega^{(l)}}(I_{(m)}, \Psi_{(i)}) + \sum_{g\in F}\frac{1}{\Omega_{lm}}(\theta_l - 1)Cov_{\Omega^{(l)}}(I_{(m)}, \Psi_{(g)})\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i},$$

$$\frac{\mathrm{d}\log\Psi_{lm}}{\mathrm{d}\log A_i} = \sum_{k\in N}(\theta_k - 1)\frac{\Psi_{lk}}{\Psi_{lm}}Cov_{\Omega^{(k)}}(\Psi_{(m)}, \Psi_{(i)}) + \sum_{g\in F}\sum_{k\in N}(\theta_k - 1)\frac{\Psi_{lk}}{\Psi_{lm}}Cov_{\Omega^{(k)}}(\Psi_{(m)}, \Psi_{(f)})\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i},$$

where I is the identity matrix.

elasticities of substitution across industries in production and in final demand are below one significantly increases estimates of gains from trade.

Real Output and Real Expenditure or Welfare

Theorem 5 gives the endogenous responses of factor shares to shocks as a function of microeconomic primitives. These were left implicit in Theorems 2 and 3. Theorem 5 can therefore be used in conjunction with Theorems 2 or 3 to characterize the response of welfare to shocks as a function of microeconomic primitives, up to the first order.

Theorem 5 also gives the endogenous responses to shocks of producer sales shares or Domar weights. Since the response of real output to productivity shocks is given by the corresponding local Domar weight, Theorem 5 can also be used to give the response of real output to shocks, up to the second order:

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log A_j}=\lambda_j^{Y_c},\quad \frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log A_j\,\mathrm{d}\log A_i}=\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log A_i}=\lambda_j^{Y_c}\left(\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log A_i}-\sum_{f\in N_c}\Lambda_f^{Y_c}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i}\right),$$

where $\mathrm{d}\,\lambda_j/\,\mathrm{d}\log A_i$ and $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_i$ are given by Theorem 5.[28]

Prices and Quantities

Armed with Theorem 5, it is straightforward to characterize the response of prices and quantities to shocks.[29]

Corollary 5. *(Prices and Quantities) The changes in the wages of factors and in the prices and quantities of goods in response to a productivity shock to producer i are given by:*

$$\begin{aligned}\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_i}&=\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i},\\ \frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_i}&=-\Psi_{ji}+\sum_{g\in F}\Psi_{jg}\frac{\mathrm{d}\log w_g}{\mathrm{d}\log A_i},\\ \frac{\mathrm{d}\log y_j}{\mathrm{d}\log A_i}&=\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log A_i}-\frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_i},\end{aligned}$$

[28]The expression for $\mathrm{d}^2\log Y_c/(\mathrm{d}\log A_j\,\mathrm{d}\log A_i)$ is a gross abuse of notation and must be handled with care. We do not dwell on the subtleties in this paper, but technically, the change in real output from one allocation to another in general depends on the *path* taken. Hulten's theorem guarantees that changes in real output are a path integral of the vector field defined by the local Domar weights along a path of productivity changes. Hence, the expression $\mathrm{d}^2\log Y_c/(\mathrm{d}\log A_j\,\mathrm{d}\log A_i)$ is really the derivative of the vector field defined by the local Domar weights. Conditional on the path taken from one allocation to the next, it can be used to compute the second derivative of the change in output at any point along that path.

[29]Recall that prices are expressed in the numeraire where $GDP = GNE = 1$ at the world level.

where $\mathrm{d}\log \Lambda_f / \mathrm{d}\log A_i$ *is given in Theorem 5.*

These results on the responses of prices and quantities to productivity shocks, and hence by implication to shocks to factor supplies and to iceberg trade costs, generalize the classic results of Stolper and Samuelson (1941) and Rybczynski (1955).[30]

5.2 Example Applications of Theorem 5

In this subsection, we show that Theorem 5 can also be used to answer questions unrelated to welfare, such as for example questions involving the aggregation of trade elasticities or structural transformation in open economies.[31,32]

Example: Aggregating and Disaggregating Trade Elasticities

We start by defining a class of aggregate elasticities. Consider two sets of producers I and J. Let $\lambda_I = \sum_{i\in I} \lambda_i$ and $\lambda_J = \sum_{j\in J}$ be the aggregate sales shares of producers in I and J, and let $\chi_i^I = \lambda_i/\Lambda_I$ and $\chi_j^J = \lambda_j/\Lambda_J$. Let k be another producer. We then define the following aggregate elasticities capturing the bias towards I vs. J of a productivity shock to m as:

$$\varepsilon_{IJ,m} = \frac{\partial(\lambda_I/\lambda_J)}{\partial \log A_m},$$

where the partial derivative indicates that we allow for this elasticity to be computed holding some things constant.

To shed light on trade elasticities, we proceed as follows. Consider a set of producers $S \subseteq N_c$ in a country c. Let J be denote a set of domestic producers that sell to producers in S, and I denote a set of foreign producers that sell to producers in S. Without loss of generality, using the flexibility of network relabeling, we assume that producers in I and

[30]See Appendix G for a discussion of how our, by taking a limit, our results can be applied to economies where traded goods are perfect substitutes as assumed by these theorems.

[31]Adao et al. (2017) show that economies of the sort that we consider can be represented as economies in which only factors are traded within and across borders, and households have preferences over factors. Theorem 5 can be used to flesh out this representation by locally characterizing its associated reduced-form Marshallian demand for factors in terms of sufficient-statistic microeconomic primitives: the expenditure share of household c on factor f is given by Ψ_{cf}; the elasticities $\partial \log \Psi_{cf}/\partial \log A_i$ holding factor prices constant then characterize its Marshallian price elasticities as well as its Marshallian elasticities with respect to iceberg trade shocks

$$\frac{\partial \log \Psi_{cf}}{\partial \log A_i} = \sum_{k\in N} \frac{\Psi_{ck}}{\Psi_{cf}} (\theta_k - 1) Cov_{\Omega^{(k)}}(\Psi_{(f)}, \Psi_{(i)}).$$

The reduced-form factor demand system is locally stable with respect to a single shock $\mathrm{d}\log A_i$ if, and only, if $\partial \log \Psi_{cf}/\partial \log A_i = 0$, with a similar conditions for a combination of such shocks.

[32]We refer the reader to Appendix J for more examples involving the factor bias of trade, showing adverse trade shocks can reduce the capital share in all countries.

J are specialized in selling to producers in S so that they do not sell to producers outside of S.

Consider an iceberg trade cost modeled as a negative productivity shock $\mathrm{d}\log(1/A_m)$ to some producer m. We then define the trade elasticity as $\varepsilon_{IJ,k} = \partial(\lambda_J/\lambda_I)/\partial\log(1/A_m) = \partial(\lambda_I/\lambda_J)/\partial\log A_m$. As already mentioned, the partial derivative indicates that we allow for this elasticity to be computed holding some things constant. There are therefore different trade elasticities, depending on exactly what is held constant. Different versions of trade elasticities would be picked up by different versions of gravity equations regressions with different sorts of fixed effects and at different levels of aggregation.

There are several possibilities for what to hold constant, ranging from the most partial equilibrium to the most general equilibrium. At one an extreme, we can hold constant the prices of all inputs for all the producers in I and J and the relative sales shares of all the producers in S:

$$\varepsilon_{IJ,m} = \sum_{s\in S}\sum_{i\in I}\chi_i^I(\theta_s-1)\frac{\lambda_s}{\lambda_i}Cov_{\Omega^{(s)}}(I_{(i)},\Omega_{(m)}) - \sum_{s\in S}\sum_{j\in J}\chi_j^J(\theta_s-1)\frac{\lambda_s}{\lambda_j}Cov_{\Omega^{(s)}}(I_{(j)},\Omega_{(m)}),$$

where $I_{(i)}$ and $I_{(j)}$ are the ith and jth columns of the identity matrix. An intermediate possibility is to hold constant the wages of all the factors in all countries:

$$\varepsilon_{IJ,k} = \sum_{i\in I}\chi_i^I\Gamma_{ik} - \sum_{j\in J}\chi_j^J\Gamma_{jk}.$$

And at the other extreme, we can compute the full general equilibrium:

$$\begin{aligned}\varepsilon_{IJ,m} = \sum_{i\in I}\chi_i^I &\left(\Gamma_{im} - \sum_{g\in F}\Gamma_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right) \\ &- \sum_{j\in J}\chi_j^J\left(\Gamma_{jm} - \sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right),\end{aligned}$$

$\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_m$ is given in Theorem 5.

The trade elasticity is a linear combination of microeconomic elasticities of substitution, where the weights depend on the input-output structure. Except at the most microeconomic level where there is a single producer s in S and in the most partial-equilibrium setting where we recover $\epsilon_s - 1$, this means that the aggregate trade elasticity is typically an endogenous object, since the input-output structure is itself endogenous.[33]

[33] In Appendix J, we provide necessary and sufficient conditions for the trade elasticity to be constant in the way.

Furthermore, in the presence of input-output linkages, it is typically nonzero even for trade shocks that are not directly affecting the sales of I to J, except in the most partial-equilibrium setting.

Example: Trade Elasticity in a Round-About World Economy

In many trade models, the trade elasticity, defined holding factor wages constant, is an invariant structural parameter. As pointed out by Yi (2003), in models with intermediate inputs, the trade elasticity can easily become an endogenous object.Consider the two-country, two-good economy depicted in Figure 2. The representative household in each country only consumes the domestic good, which is produced using domestic labor and imports with a CES production function with elasticity of substitution θ. We consider the imposition of a trade cost hitting imports by country 1 from country 2. For the sake of illustration, we assume that the trade cost does not apply to the exports of country 1 to country 2.

The trade elasticity holding factor wages constant is given by

$$\frac{\theta - 1}{1 - \Omega_{21}\Omega_{12}},$$

where Ω_{ij} is the expenditure share of i on j, e.g. its intermediate input import share. As the intermediate input shares increase, the trade elasticity becomes larger. Simple trade models without intermediate goods are incapable of generating these kinds of patterns.

Of course, since the intermediate input shares Ω_{ij} are themselves endogenous (depending on the iceberg shock), this means that the trade elasticity varies with the iceberg shocks. In particular, if $\theta > 1$, then the trade elasticity increases (nonlinearly) as iceberg costs on imports fall in all countries since intermediate input shares rise. [34]

Example: Baumol's Cost-Disease and Export-Led Growth

We illustrate the nonlinear effects of trade and productivity shocks on real output discussed in Section 5.1 via a simple example showing how opening up to trade and relying on export-led growth can overcome Baumol's cost disease. Baumol's cost disease is a phenomenon whereby in the presence of complementarities across sectors, the relative sales of sectors with relatively faster productivity growth shrink over time as a result of

[34] In Appendix J we show that there it is possible to generate "trade re-switching" examples where the trade elasticity is non-monotonic with the trade cost (or even has the "wrong" sign) in otherwise perfectly respectable economies. These examples are analogous to the "capital re-switching" examples at the center the Cambridge Cambridge Capital controversy.

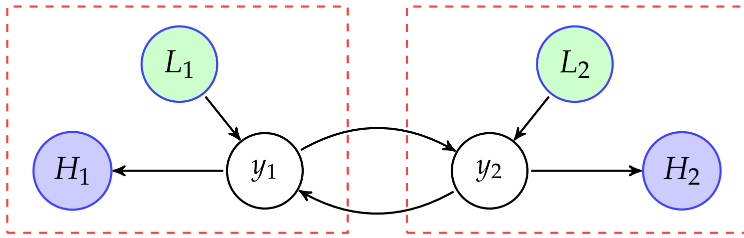


Figure 2: The solid lines show the flow of goods. Green nodes are factors, purple nodes are households, and white nodes are goods. The boundaries of each country are denoted by dashed box.

their higher productivity growth. It reduces down the growth rate of aggregate productivity over time. As discussed in Baqaee and Farhi (2017a), Baumol's cost disease is a manifestation of nonlinearities.

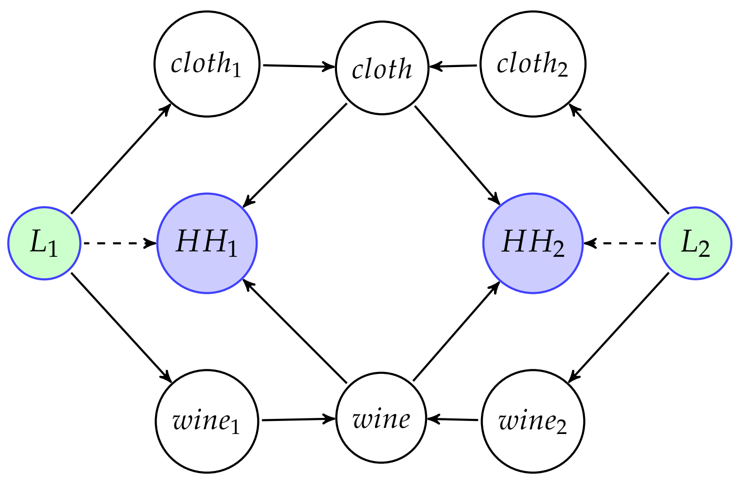


Figure 3: The solid lines show the flow of goods while the dashed lines show the flow of wage payments.

Consider the economy depicted in Figure 3. Countries 1 and 2 produce varieties of wine and cloth. The representative household in each country consumes a composite of foreign and domestic varieties of wine and cloth. We assume that the elasticity of substitution across foreign and domestic varieties of wine or cloth is $\theta > 1$, but that the elasticity of substitution between wine and cloth is $\sigma < 1$. To simplify the algebra, assume that there is no home-bias, so that both households consume the same basket of wine and cloth. Finally, we assume that wine and cloth have the same size at the initial point so that $\lambda^Y_{cloth} = \lambda^Y_{wine} = 1/2$ and $\lambda^{Y_1}_{cloth_1} = \lambda^{Y_1}_{wine_1} = 1/2$. The relative size of country 1 at the initial point is χ^Y_1. It is also the share of country 1's varieties in the overall baskets of wine and cloth. It therefore also indexes the degree of openness of country 1: when $\chi^Y_1 = 1$, it is a closed world economy; when $\chi^Y_1 = 0$, country it is a small open economy.

The effect on the real output of country 1 from an increase $\Delta log A_{cloth_1}$ in the productivity of its cloth is given up to the second order by[35]

$$\Delta \log Y_1 \approx \log \frac{\mathrm{d} \log Y_1}{\mathrm{d} \log A_{cloth_1}} \Delta \log A_{cloth_1} + \frac{1}{2} \frac{\mathrm{d}^2 \log Y_1}{\mathrm{d} \log A^2_{cloth_1}} (\Delta \log A_{cloth_1})^2,$$

with

$$\frac{\mathrm{d} \log Y_1}{\mathrm{d} \log A_{cloth_1}} = \lambda^{Y_1}_{cloth_1} = \frac{1}{2}, \quad \frac{\mathrm{d}^2 \log Y_1}{\mathrm{d} \log A^2_{cloth_1}} = \frac{\mathrm{d}\, \lambda^{Y_1}_{cloth_1}}{\mathrm{d} \log A_{cloth_1}} = \frac{(\chi^Y_1)^2(\sigma - 1)}{2} + \frac{(1-\chi^Y_1)(\theta-1)}{4}.$$

The second-order term capture the extent to which large shocks have larger or smaller proportional effects than small shocks, conditional on the size of the sector. Cumulating rates of productivity growth over time is equivalent to increasing the size of the shock. The second-order term therefore captures the strength of Baumol's cost disease. When it is negative, Baumol's cost disease obtains. When it is positive, we have a form of reverse Baumol's cost disease where the sector with faster productivity growth expands instead of shrinking.

As we increase the relative size and openness χ^Y_1 of country 1, the effect on its real output of the productivity of its cloth becomes smaller because Baumol's cost disease becomes stronger. In the small-open economy limit $\chi^Y_1 \to 0$, we have $\mathrm{d}^2 \log Y_1 / \mathrm{d} \log A^2_{cloth_1} = (\theta - 1)/4 > 0$, and so there is reverse Baumol's cost disease. In the large-closed-economy limit $\chi^Y_1 \to 1$, we have $\mathrm{d}^2 \log Y_1 / \mathrm{d} \log A^2_{cloth_1} = (\sigma - 1)/2 < 0$ and so there is Baumol's cost disease.

Of course, if country 1 is small and closed, then it is as if it were a large-closed economy. Opening up to international trade turns it into a small-open economy. Trade can therefore overcome (and indeed overturn) Baumol's cost-disease by allowing export-led growth.

6 Tariffs and Other Distortions

So far, we have maintained the assumption of frictionless competitive markets. In this section, we extend our results to allow for tariffs, or more generally for distortions that can be modeled as explicit or implicit taxes.

We start by summarizing how to generalize the ex-post and ex-ante comparative statistic results of Sections 3 and 5 in the presence of large tariffs or other distortions. As far

[35] Here again, we slightly abuse notation by using derivative symbols since Y_1 is not a function.

as we are aware, this is the first time such comparative statics have been derived in the literature, and so we consider them to be an important contribution of this paper. However, in order to avoid repetition, we relegate the formal statement of the results and the underlying analysis to Appendix E.

We focus instead on a different perspective showing that the losses from small tariffs or other distortions are given, up to a second order, by a Domar-weighted sum of deadweight-loss (or Harberger) triangles (Harberger, 1964). As usual, we present our result in two ways, using ex-post and ex-ante sufficient statistics.

6.1 Allowing for Distortions

We denote by μ the $N \times 1$ matrix of tax wedges, where the ith element is an ad valorem tax on the output of producer i. To state our results, we assume that the revenue generated by the wedge μ_i are *included* in the revenue of producer i (the producer collects the tax revenue as part of its revenue, and then pays the government). This is merely an accounting convention, and it is straightforward to convert our results for situations where the revenue generated by the wedge are not included in revenues. We can capture tariffs on imports (exports) as taxes on specialized importers (exporters) who buy domestic (foreign) goods and sell them across borders. These wedges can also capture other distortions such as markups or financial constraints.

6.2 Generalizing the Comparative Statics in Sections 3 and 5

Theorems 1, 2, 3, and 5 are generalized to allow for arbitrary distortions in Appendix E. We characterize the ex-ante and ex-post comparative static effects of shocks to productivities, factor supplies, iceberg trade costs, and taxes, in the presence of large pre-existing tariffs or other distortions.

The analysis reveals important and interesting differences. For example, in the presence of distortions such as tariffs or markups, shocks to productivities, factor supplies, and iceberg trade costs outside of a country now typically have first-order effects on its real output and on its aggregate productivity. As a result, the Hulten-like result for real output fails. This is because there are now non-trivial first-order reallocation effects in real output. It follows that the Hulten-like results for welfare of Corollary 1 also fail.[36] Even when these results fail, our output- and welfare-accounting results can be general-

[36] Case (ii) of Corollary 1 (Cobb-Douglas Hulten-like result) for welfare continues to hold for shocks to productivities, factor supplies, and iceberg trade costs, because the absence of reallocation effects is preserved.

ized by characterizing the new corresponding reallocation effects as a function of changes in distorting wedges and changes in factor income shares. These changes in factor income shares are in turn characterized as functions of microeconomic primitives by generalizations of our propagation equations.

6.3 Costs of Tariffs and Other Distortions: Ex-Post Sufficient Statistics

We write tariffs or other distortions as $\exp(\Delta \log \mu_i)$ and provide approximations for small tariffs $\Delta \log \mu_i$ around an efficient equilibrium with no tariffs or other distortions.

Real Output

We start by characterizing changes in real output.

Theorem 6 (Reduced-Form Output Loss)**.** *Starting at an efficient equilibrium, up to the second order, in response to the introduction of small tariffs or other distortions:*

(i) changes in world real output and real expenditure are given by

$$\Delta \log Y = \Delta \log W \approx \frac{1}{2} \sum_{i \in N} \lambda_i \Delta \log y_i \Delta \log \mu_i;$$

(ii) changes in the real output of country c are given by:

$$\Delta \log Y_c \approx \frac{1}{2} \sum_{i \in N_c} \lambda_i^{Y_c} \Delta \log y_i \Delta \log \mu_i.$$

Hence, for both the world and for each country, the reduction in real output from tariffs and other distortions is given by the sum of all the deadweight-loss triangles $1/2 \Delta \log y_i \Delta \log \mu_i$ weighted by their corresponding local Domar weights. Harberger (1964) had shown that the classic welfare deadweight-loss triangle intuition from partial equilibrium models could be applied to general equilibrium models to measure changes in welfare in the presence of income effects and with compensating transfers. Theorem 6 shows that it can be used in general equilibrium, in the presence of income effects, and in the absence of compensating transfers, to measure changes in real output. To the best of our knowledge this is a new result in growth-accounting.

Conditional on matching both the Domar weights λ_i and $\lambda_i^{Y_c}$ and the changes in the quantities $\Delta \log y_i$ of all goods, the details of the production structure are irrelevant. In particular, conditional on these sufficient statistics, we do not need to know anything about whether or not there are international (or domestic) production networks. However, as we shall show, input-output linkages do affect Domar weights and changes in the quantities of all goods, and so accounting for input-output linkages does matter. As we shall see, accounting for global value chains matters a great deal for the quantitative effects of tariffs: the triangles $1/2\Delta\log y_i \Delta\log\mu_i$ are larger, and they are also aggregated with larger weights given by sales shares λ_i and $\lambda_i^{Y_c}$rather than value-added shares.[37]

To give some intuition for Theorem 6, we focus on the country level result for simplicity. Starting at an efficient equilibrium, the introduction of tariffs or other distortions leads to changes $\Delta\log y_i$ in the quantities of goods $i \in N_c$ in country c and to changes in the wedges $\Delta\log\mu_i$ between prices and marginal costs. The price-cost margin $p_i\Delta\log\mu_i$ measure the wedge between the marginal contribution to country real output and the marginal cost to real output of increasing the quantity of good i by one unit. Hence, $\lambda_i^{Y_c}\Delta\log\mu_i$ is the marginal proportional increase in real output from a proportional increase in the output of good i. Integrating from the initial efficient point to the final distorted point, we find that $(1/2)\lambda_i^{Y_c}\Delta\log y_i\Delta\log\mu_i$ is the contribution of good i to the change in real output.

Changes $\Delta\log y_i$ in the quantities of goods $i \in N_c$ in country c can be driven by changes in tariffs or other distortions in the country or in other countries. This shows that changes in tariffs or other distortions outside of the country affect the aggregate productivity of the country. This is in a sharp contrast to the results that we derived earlier for efficient economies.

It is instructive to compare the costs of tariffs to the costs of an increase in iceberg costs. In response to a change in iceberg costs outside of the country , the change in country real output is zero at any order of approximation. At the world level, in response to a change $\Delta\log(1/A_i)$ in iceberg trade costs, the change in real output or real expenditure is given up to a second-order by the sum of trapezoids rather than triangles:

$$\Delta\log Y = \Delta\log W \approx -\sum_{i\in N}\lambda_i\left(1+\frac{1}{2}\Delta\log\lambda_i\right)\Delta\log(1/A_i).$$

In contrast to equivalent shocks to tariffs, the first-order effect of shocks iceberg trade costs have nonzero first-order effects. This is a way to see why the costs of non-tariff

[37]This partly affirms Paul Krugman's view that trade in intermediate inputs does not alter the basic qualitative logic of why trade barriers are harmful. Even in models with global value chains, tariffs are harmful because they cause deadweight losses captured by Harberger trianles. See *Does Trade in Intermediate Goods Alter the Logic of Costs From Protectionism?* by Paul Krugman (2018). However, it does turn out to alter the quantitative message though: in models with global value chains, the triangles are larger, and for a given set of triangles, they are more important.

trade barriers are typically so much higher than tariff trade barriers in trade models.

Real Expenditure and Welfare

Theorem 6 shows how real output responds to changes in tariffs or other distortions. These results do not apply to welfare. At the country level, changes in tariffs and other distortions typically lead to first-order changes (due to terms of trade/reallocation effects). But even at the world level where these effects wash out, changes in real expenditure no longer coincide with changes in welfare, since changes in world real expenditures $d \log W$ cannot be integrated to arrive at a well-defined social welfare function.[38]

To proceed, we introduce a homothetic Bergson-Samuelson social welfare function

$$W^{BS}(W_1, \ldots, W_C) = \sum_c \overline{\chi}_c^W \log W_c,$$

where $\overline{\chi}_c^W$ is the initial income share of country c at the efficient equilibrium. These welfare weights are chosen so that there is no incentive to redistribute across agents at the initial equilibrium. Even though this welfare function has no desire to redistribute across agents at the initial point, distributive effects across households do appear at the second-order in response to shocks. We did not need to concern ourselves with this second-order effect in our first-order characterizations for efficient economies in Sections 3 and 5, and the same goes for our first-order characterization of country welfare in this section.

We compute the change in welfare from the introduction of tariffs or other distortions in terms of the proportional increase in the consumption of all goods by all households at the final distorted equilibrium which would keep the welfare function constant at the level of the initial efficient steady state. Formally, we measure changes in welfare by $\Delta \log \delta$, where δ solves the equation

$$W^{BS}(\overline{W}_1, \ldots, \overline{W}_C) = W^{BS}(W_1/\delta, \ldots, W_C/\delta),$$

where $\overline{W}_c$ and W_c are the values at the initial efficient equilibrium. We use a similar definition for country level welfare δ_c

Corollary 6 (Reduced-Form Welfare)**.** *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions:*

[38]This has to do with the fact that individual household preferences across all countries are non-aggregable.

(i) changes in world welfare are given up to the second order by

$$\Delta \log \delta \approx \Delta \log W + Cov_{\chi_c^W}\left(\Delta \log \chi_c^W, \Delta \log P_{W_c}\right);$$

(ii) changes in country real expenditure or welfare are given up to the first order by

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c}.$$

The change in world welfare is the sum of the change in world real expenditure (output) and a redistributive term. The redistributive term is positive whenever the covariance between the changes in household income shares and the changes in consumption price deflators is positive. It captures a familiar deviation from perfect risk sharing. It would be zero if households could engage in perfect risk sharing before the introduction of the tariffs or other distortions. In our applications, this redistributive effect is quantitatively small and so changes in world welfare are approximately equal to changes in world real output.

6.4 Costs of Tariffs and Other Distortions: Ex-Ante Sufficient Statistics

Theorem 6 and Corollary 6 express the effects of tariffs and other distortions in terms of endogenous individual output changes up to the second order. In this subsection, we provide formulas for these individual output changes, and hence for the effects of tariffs and other distortions, in terms of primitives: microeconomic elasticities of substitution and the HAIO matrix.

Our results will make use of the following characterizations. Changes in factor shares are given up to the first order by the system of linear equations

$$\begin{aligned}\Delta \log \Lambda_f \approx -\sum_{i\in N}\Gamma_{fi}\Delta \log \mu_i - \sum_{g\in F}\Gamma_{fg}\Delta \log \Lambda_g + \sum_{g\in F}\Xi_{fg}\Delta \log \Lambda_g \\ - \sum_{i\in N}\frac{\lambda_i}{\Lambda_f}\Psi_{if}\Delta \log \mu_i + \sum_{i\in N}\Xi_{fi}\Delta \log \mu_i,\end{aligned}$$

where the definition of Ξ is extended for $f \in F$ and $i \in N$ by $\Xi_{fi} = \frac{1}{\Lambda_f}\sum_{c\in C}(\Lambda_f^{W_c} - \Lambda_f)\Phi_{ci}\lambda_i$, and Φ_{ci} is the share of the revenue raised by the tariff or other distortion on good i which accrues to country c. Changes in country income shares are given up to the first order by

$$\chi_c^W \Delta \log \chi_c^W \approx \sum_{g\in F}\Phi_{cf}\Lambda_g \Delta \log \Lambda_g + \sum_{i\in N}\Phi_{ci}\lambda_i \Delta \log \mu_i.$$

Changes in country real expenditure deflators are given up to the first order by

$$\Delta \log P_{W_c} \approx \sum_{i\in N} \lambda_i^{W_c} \Delta \log \mu_i + \sum_{g\in F} \Lambda_g^{W_c} \Delta \log \Lambda_g.$$

In these equations, changes in tariffs and other distortions play three distinct roles. First, as described in Section 5, they act via prices like negative productivity shocks, and the changes in factor wages that they trigger also have similar effects. Second, they raise revenues. Third, for given sales and revenues, and for given factor wages, they reduce input (and hence ultimately factor) demand.

The system of linear equations for the changes in factor income shares $\Delta \log \Lambda_f$ is similar to the one that we encountered in Section 5, with two new terms on the second line, $-\sum_{i\in N} \frac{\lambda_i}{\Lambda_f} \Psi_{if} \Delta \log \mu_i$ representing reductions in factor demand, and $\sum_{i\in N} \Xi_{fi} \Delta \log \mu_i$ representing distribution effects arising from differential ownerships across households with different expenditure patterns of the revenues raised by the tariffs or other distortions. The formula for the changes in the country income shares $\chi_c^W \Delta \log \chi_c^W$ is also similar to the one that we encountered in Section 5, with a new second term $\sum_{i\in N} \Phi_{ci} \lambda_i \Delta \log \mu_i$ representing the part of the revenue raised by the tariff or other distortion on good j which accrues to country c. The formula for the changes in the real expenditure deflators $\Delta \log P_{W_c}$ is again similar to the one that that we encountered in Section 5.

The main results of this section, Theorem 7 (real output) and Corollary 7 (welfare) below, build on these structural characterizations of changes in factor shares, country income shares, and country price deflators.

Theorem 7 (Structural Output Loss)**.** *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions:*

(i) changes in world real output and real expenditure are given up to the second order by

$$\begin{aligned}\Delta \log Y = \Delta \log W \approx &-\frac{1}{2} \sum_{l\in N} \sum_{k\in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j\in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}) \\ &- \frac{1}{2} \sum_{l\in N} \sum_{g\in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j\in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}) \\ &+ \frac{1}{2} \sum_{l\in N} \sum_{c\in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l);\end{aligned}$$

(ii) changes in the real output of country c are given up to the second order by

$$\Delta \log Y_c \approx -\frac{1}{2}\sum_{l\in N_c}\sum_{k\in N}\Delta \log \mu_k \Delta \log \mu_l \sum_{j\in N}\lambda_j^{Y_c}\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)})$$
$$-\frac{1}{2}\sum_{l\in N_c}\sum_{g\in F}\Delta \log \Lambda_g \Delta \log \mu_l \sum_{j\in N}\lambda_j^{Y_c}\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)})$$
$$+\frac{1}{2}\sum_{l\in N_c}\sum_{c\in C}\chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c}-\lambda_l)/\chi_c^Y.$$

For brevity, we only discuss changes in real output $\Delta \log Y$. First, all the terms scale with the square of the tariffs or other distortions $\Delta \log \mu$. There is therefore a sense in which misallocation increases with the tariffs and other distortions. Second, all the terms scale with the elasticities of substitution θ of the different producers. There is therefore a sense in which elasticities of substitution magnify the costs of these tariffs and other distortions. Third, all the terms also scale with the sales shares λ of the different producers and with the square of the Leontief inverse matrix Ψ. There is therefore also a sense in which accounting for intermediate inputs magnifies the costs of tariffs and other distortions. Fourth, all the terms mix the tariffs and other distortions, the elasticities of substitution, and of properties of the network. Hence, in general, the costs of tariffs and other distortions depends on how they are distributed over the network.

For a given producer $l \in N$, there are terms in $\Delta \log \mu_l$ on the three lines. Taken together, these terms sum up to the Harberger triangle $(1/2)\lambda_l \Delta \log \mu_l \Delta \log y_l$ corresponding to good l in terms of microeconomic primitives. The three lines break it down into three components, corresponding to three different effects responsible for the change in the quantity $\Delta \log y_l$ of good l.

The term $-(1/2)\sum_{k\in N}\Delta \log \mu_k \Delta \log \mu_l \sum_{j\in N}\lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)})$ on the first line corresponds to the change $\Delta \log y_l$ in the quantity of good l coming from *substitutions* by all producers j in response to changes in all tariffs and other distortions $\Delta \log \mu_k$, holding factor wages constant.

The term $-(1/2)\sum_{g\in F}\Delta \log \Lambda_g \Delta \log \mu_l \sum_{j\in N}\lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)})$ on the second line corresponds to the change $\Delta \log y_l$ in the quantity of good l coming from *substitutions* by all producers j in response to the endogenous changes in factor wages $\Delta \log w_g = \Delta \log \Lambda_g$ brought about by all the changes in tariffs and other distortions.

The term $\sum_{c\in C}\chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c}-\lambda_l)$ on the third line corresponds to the change $\Delta \log y_l$ in the quantity of good l coming from *redistribution* across agents with different spending patterns, in response to the endogenous changes in factor wages brought about by all the changes in tariffs and other distortions.

Note that in contrast to the expressions for changes in sales shares, the two substitution terms feature, for each producer j, the elasticity of substitution θ_j and not $\theta_j - 1$. This is because they characterize the change in the *quantity* of good l, and not the change in its *sales*. The neutral case leading to no changes in no longer Cobb Douglas but Leontief: when all the elasticities of substitution are zero, there are no changes in the quantity of good l, and no effect of tariffs or other distortions on real output.

Corollary 7 (Structural Welfare)**.** *Starting at an efficient equilibrium, changes in world and country welfare $\Delta \log \delta$ and $\Delta \log \delta_c \approx \Delta \log W_c$ are given via Corollary 6, respectively up to the second order (world) and up to the first order (country).*

Example: Tariffs in a Round-About World Economy

Consider again the example in Figure 2. Suppose that the two countries are symmetric, and that each country introduce a symmetric tax $\Delta \log \mu$ on its imports from the other country. Because of symmetry, changes in country real output, country real expenditure or welfare, changes in world real output changes in world real expenditure or welfare, are all the same. Hence, using Theorem 7, the effects of the tariffs on any of these variables is given by

$$\frac{1}{2}(\lambda_{12}\Delta \log y_{12}\Delta \log \mu + \lambda_{21}\Delta \log y_{21}\Delta \log \mu) = -\theta \frac{\Omega}{2(1-\Omega)^2}(\Delta \log \mu)^2,$$

where y_{12} is the quantity of imports from country 2 by country 1, λ_{12} is the corresponding sales share, and y_{21} and λ_{21} are defined similarly. Because of symmetry $y_{12} = y_{21}$ and $\lambda_{12} = \lambda_{21}$.

The losses increase with the elasticity of substitution θ and with intermediate input share Ω. This is both because the relevant sales shares $\lambda_{12} = \lambda_{21} = \Omega/[2(1-\Omega)]$ become larger and because the reductions in the quantities of imports $-\Delta \log y_{12} = -\Delta \log y_{21} = [\theta/(1-\Omega)]\Delta \log \mu$ become larger. The latter effect occurs because when the intermediate input share increases, goods effectively cross borders more times, and hence get hit by the tariffs more times, which increases the relative price of imports more and leads to a larger reduction in their quantity.

7 Applications

In this section, we conduct some empirical applications of our results. We use a structural multi-factor production network model, calibrated to fit world input-output data. We quantify the way trade costs (tariffs or iceberg) affect output, welfare, and factor rewards.

We consider three suggestive scenarios which relate to recent events: (1) a universal change in (tariff or non-tariff) barriers, including removal of all existing tariffs; (2) an increase of trade barriers between Great Britain and the European Union; and (3) an increase in trade barriers between the United States and China. The numbers are interesting by themselves, but more importantly for us, they demonstrate and quantify some of the mechanisms that we have emphasized analytically. These applications also help us illustrate the importance of global input-output linkages. To bring this point out, we compare the results of the benchmark model to alternative calibrations which trivialize the input-output connections in different ways commonly used in the trade literature.

We end the section by applying our welfare-accounting formulas to perform our terms-of-trade and reallocation decompositions of changes in real expenditure in different countries.

7.1 Counterfactuals

We start by presenting our benchmark calibration with input-output linkages and our alternative value-added calibrations. We then put them to work on our three counterfactual scenarios.

Benchmark Calibration with Input-Output Linkages

The benchmark model consists of 40 countries as well as a "rest-of-the-world" composite country, each with four factors of production: high-skilled, medium-skilled, low-skilled labor, and capital. Each country has 30 industries each of which produces a single industry good. Due to lack of data, we assume that there is a representative household in each country, and that this household only derives income from domestic factors. Unbalanced trade is captured via international transfers.

The model has a nested-CES structure. Each industry produces output by combining its value-added (consisting of the four domestic factors) with intermediate goods (consisting of the 30 industries). The elasticity of substitution across primary factors is γ, and across intermediates is ζ, the elasticity of substitution between factors and intermediate inputs is θ, the elasticity of substitution in consumption goods across industries is σ. When a producer or the household i in country c purchases inputs of some industry j, it consumes a CES aggregate of goods from this industry sourced from various countries with some trade elasticity ε_j. We use data from the WIOD to calibrate the CES share parameters to match expenditure shares in the year 2008.

To calibrate the elasticities of substitution, we use the same elasticities as in Section

4. That is, we set the elasticity of substitution across industries $\zeta = 0.2$, the one between value-added and intermediates $\theta = 0.5$, the elasticity of substitution in consumption $\sigma = 0.9$, and the trade elastiities ε_j following the estimates from Caliendo and Parro (2015). Finally, we set the elasticity of substitution among primary factors $\gamma = 0.5$. See Appendix A for more details.

Value-Added Calibrations

The benchmark model features input-output linkages. To emphasize the importance of taking input-output lnkages into account and to connect with our analytical results, we compare the benchmark model with input-output linkages to two alternative value-added calibrations which are common in the trade literature. These alternative calibrations both assume that all production takes place with value-added production functions (no intermediates) but trivialize the input-output connections in two different ways. We call these two calibrations the low-trade value-added (LVA) and the high-trade value-added (HVA) economies. As we shall see, these two value-added calibrations are problematic, because they are not exact representations of the benchmark economy.[39]

Low-trade value-added (LVA): the value-added produced by each producer matches the one in the data. It is then assumed that the fraction of the value added of i which is sold to each country is equal to the corresponding fraction of the sales of i in the data. We call this the low-trade value-added economy because it reduces the overall value of trade as a share of GDP. This is the procedure used by Costinot and Rodriguez-Clare (2014) in the handbook chapter for their value-added calibration, and it is also how ACR mapped their model to the data.

High-trade value-added (HVA): the value-added produced by each producer matches the one in the data. However, it is then assumed that the value of i which is sold to each foreign country is equal to the corresponding sales of i in the data. The residual value-added is sold in the domestic country of i. We call this the high-trade value-added

[39]The only correct way of representing this economy with intermediate inputs as a value-added economy is to follow Adao et al. (2017) by assuming that only factors are traded within and across borders, and that households have preferences over factors. For some welfare counterfactuals, this representation can be put to use by directly specifying and estimating a parsimoniously-parametrized factor demand system. This parsimony advantage must of course be traded of against the cost of misspecification, and, as our formulas in Section 5 make clear, the "true" functional form of the factor demand system is likely to be complex for realistic economies. Furthermore, using this approach is more difficult in the presence of intermediate goods, trade costs, and tariffs. For example, shocks to iceberg costs of trading goods between two countries must first be translated into shocks to costs of trading the factors used directly and indirectly to produce them, which must be handled by expanding the set of factors to reflect the fact that the same factor may be crossing boundaries a different number of times along different global supply chains. These difficulties are compounded in the presence of tariffs.

	AUS	BEL	BGR	CAN	CHN	DEU	IRL	LUX	USA	World
Benchmark Iceberg	-2.0%	-6.7%	-5.9%	-3.3%	-1.3%	-3.0%	-8.3%	-16.9%	-1.1%	-2.26%
HVA Iceberg	-2.1%	-5.2%	-5.5%	-3.9%	-1.5%	-3.4%	-7.1%	-10.8%	-1.0%	-2.34%
LVA Iceberg	-0.8%	-2.3%	-1.9%	-1.4%	-0.5%	-1.2%	-3.6%	-5.0%	-0.3%	-0.86%
Benchmark Tariff	-0.7%	-2.3%	-0.7%	-1.2%	-0.2%	-0.9%	-3.7%	-5.4%	0.1%	-0.43%
HVA Tariff	-0.5%	-0.6%	-0.1%	-1.5%	-0.1%	-0.7%	-2.1%	-5.3%	0.3%	-0.23%
LVA Tariff	-0.4%	-0.3%	0.3%	-0.7%	0.1%	-0.5%	-1.3%	-0.7%	0.1%	-0.17%

Table 2: Percentage change in real income for a subset of the countries in response to a universal 10% change in iceberg trade costs or import tariffs. We compare the results from the benchmark economy with intermediate goods and input-output linkages with economies that assume only value-added production functions (HVA and LVA).

economy because it preserves the volume of trade relative to GDP.

Scenario I: Universal Trade Shocks

This scenario illustrates the importance of taking into account input-output connections and the importance of modeling the difference between iceberg shocks and tariffs. In Table 2, we report the impact on the welfare of a few countries, as well as the effect on world welfare, of a 10% increase in either the iceberg costs of trade or a 10% increase in import tariffs. We compare the response of our benchmark economy with intermediate goods to those of the LVA and HVA economies with no intermediate goods.

Across the board, and as suggested by the discussion in Section 6.3, an increase in iceberg trade costs (or other non-tariff barriers to trade) is significantly more costly than an equivalent increase in tariffs. For example, US welfare actually increases by 0.1% in response to increases in tariffs, but decreases by 1.1% in response to increases in trade costs. World welfare decreases by 0.4% in response to increases in tariffs, but decreases by 2.3% in response to increases in trade costs. Hence, drawing inferences about increases in tariffs by studying increases in iceberg trade costs, as sometimes happens in the literature, can be highly misleading.

Next, we compare the benchmark model to the value-added economies. The reduction in world welfare from increases in iceberg trade costs is 2.3% for the benchmark economy, it is also 2.3% for the HVA economy, but it is only 0.9% for the LVA economy. The HVA economy does a better job than the LVA economy because it preserves the volume of trade, and hence, by Theorems 1, 2, and 3, the response of world welfare in that model is, at the first order, identical to that of the benchmark model.[40] The response of country

[40]That means, as long as the shocks are sufficiently small (ruling out nonlinearities), we should expect the benchmark and HVA economies to deliver similar welfare results for the world as a whole.

welfare is different at the first order, but for the shock that we consider, these differences seem to be relatively small for most (but not all) countries. The LVA economy, which is the much more common calibration in the literature, is hopeless. Since LVA reduces the volume of trade to GDP, it greatly understates, at the first order, the welfare effects of shocks to iceberg trade costs.

The reduction in world welfare from an increase in tariffs is 0.43% for the benchmark economy, but it is only 0.23% for the HVA economy, and it is even less at 0.17% for LVA economy. In this case, neither the LVA nor the HVA economy does a good job of replicating the benchmark model.

Table 2 shows that worldwide losses from iceberg shocks in the HVA economy are approximately equal to the ones in the benchmark, whereas the LVA losses are much smaller. On the other hand, for tariffs, both the LVA and HVA economies perform badly.

Theorems 1 and 6 help shed light on why this happens. According to Theorem 1 losses in GDP from iceberg shocks are given by the Domar weight of exports multiplied by the iceberg shock $d \log \tau$. Since the HVA calibration matches the volume of trade as a share of GDP, this means that the HVA economy's response is the same as the benchmark model up to a first order. This does not hold for the LVA economy which shrinks the volume of trade relative to GDP.

According to Theorem 6, the losses from tariffs are given by $(1/2) \sum_i \lambda_i \Delta \log y_i \times \log \mu_i$, where λ_i is the sales share, $\log y_i$ is the quantity, and $\log \mu_i$ is the (gross) tax for good i. Since the HVA economy preserves the volume of trade, λ_i are the same for the benchmark and the HVA economy. Nevertheless, the response of the HVA economy is half that of the benchmark. This is because in the HVA economy, the reduction in export quantities $\Delta \log y_i$ in response to tariffs is significantly lower. The LVA economy is still hopeless, since it gets both the output elasticity $\Delta \log y_i$ wrong and the trade volumes λ_i wrong.

There are two reasons why the HVA economy underestimates the reduction in exports $\Delta \log y_i$ in response to tariffs. First, the impact of the tariffs on the price of traded goods is smaller. In the true economy, the universal tariff affects global value chains, and the tariffs are compounded each time unfinished goods cross borders, as in the round-about example of Section 6.4. This effect is necessarily absent in HVA where unfinished goods only cross borders once. Second, in the HVA economy, imported goods are a larger share of each agent's basket. This mechanically lowers the elasticity of that agent's demand for imported goods since increases in the price of imported goods increase the overall price index by more (reducing substitution away from imports).

Conceptually, one way to separate the double-marginalization effect (where tariffs are paid multiple times) from the elasticity effect is to tax traded goods based only on the domestic content of their exports.[41] For the HVA and LVA economies, since each good crosses a border at most once, this would have no effect. However, for the benchmark economy, taxing only the domestic content of traded goods would reduce global output by -0.31% instead of the benchmark -0.43%. This suggests that is a significant degree of re-exporting in world trade, but since the loss of 0.31% is still higher than the 0.23% losses in the HVA, it suggests both the elasticity effect and the double marginalization effect are quantitatively important.

Intuition from a Round-about Economy

To see these two mechanisms in more detail, we formally work through the round-about economy depicted in Figure 2. In Section 6, we showed that the output losses from tariffs in that economy where given by

$$(1/2)\sum_i \lambda_i \Delta \log y_i \times \log \mu_i = \underbrace{\frac{\Omega}{2(1-\Omega)}}_{\lambda_i} \underbrace{\frac{\theta}{(1-\Omega)}}_{\mathrm{d}\log y_i},$$

where i indexes the traded good, Ω is the share of traded goods in sales, and θ is the elasticity of substitution between traded and non-traded goods. The expression above follows from the fact that

$$\mathrm{d}\log y_i = -\theta(\mathrm{d}\log p_i) = -\frac{\theta}{1-\Omega}.$$

The term $1/(1-\Omega)$ captures the fact that global value chains amplify the effect of the tariff on the price — each time the good crosses the border, the tariff must be paid again.

Now, imagine that we take data from this economy and calibrate it using the HVA and LVA structures. For clarity, assume that the elasticity of substitution between and within sectors is the same and equal to θ. Then, for the HVA economy, we have

$$\mathrm{d}\log y_i = -\theta(\mathrm{d}\log p_i - \mathrm{d}\log P_c) = -\theta(1-\lambda_i),$$

where P_c is the consumer price index. Hence, the losses in output for the HVA economy

[41] Formally, for each traded good i produced in country c, define $\Omega^c_{ij} = \Omega_{ij}\mathbf{1}(i \in c)$ and $\Psi^c = (I - \Omega^c)^{-1}$. Let $\delta_i = \sum_{j \notin c} \Psi^c_{ij}$ and impose the tax $\mu_i = (1 + 0.10^{1-\delta_i})$ on each i. If the traded good i does not rely on foreign inputs in its supply chain, then $\delta_i = 0$, if the traded good i contains no domestic value-added (directly or indirectly) then $\delta_i = 1$.

are given by

$$-\lambda_i \underbrace{\theta(1-\lambda_i)}_{\mathrm{d}\log y_i} = -\frac{\Omega}{2(1-\Omega)} \underbrace{\theta\left(1-\frac{\Omega}{2(1-\Omega)}\right)}_{\mathrm{d}\log y_i},$$

whereas the ones for the LVA economy are

$$-\frac{\Omega}{2} \underbrace{\theta\left(1-\frac{\Omega}{2}\right)}_{\mathrm{d}\log y_i},$$

From this simple expression, we can see both effects operating. First, holding fixed the consumer price index, the effect of the tariff on the price of the traded good in both the HVA and the LVA economy is 1 instead of $1/(1-\Omega)$ – this reflects the fact the global value chains amplify the effect of the tax on the price. Secondly, the higher is the volume of trade λ_i the lower is $d\log y_i$, since when λ_i is high, the domestic price index moves by more in response to trade shocks, effectively reducing the elasticity of output with respect to the tax. Since the losses are quadratic in the volume of trade to GDP, we can see why both the HVA and LVA give broadly similar results (and both undershoot the benchmark).

If we consider taxing exporters only on the domestic content of their output for the round-about economy, instead of a $\mathrm{d}\log\mu$ tax, we would impose a tax of $(1-\Omega)\,\mathrm{d}\log\mu$. The losses on output would then be $\theta\lambda_i\,\mathrm{d}\log\mu$ instead of $\lambda_i\theta/(1-\Omega)\,\mathrm{d}\log\mu$ for the roundabout economy, whereas, in the HVA and LVA economies, we would get the exact effect as before (both of which still undershoot the roundabout model, even when though there is no double-marginalization).

Scenario II: Brexit

We consider the effect of a 10% increase in icerberg tade costs or tariffs of British goods into the EU. For this scenario, we do not delve into the details, and instead focus on some of the distributional consequences for the different factors. The detailed results can be found in in Appendix B: in Table 5 for iceberg trade costs and in Tables 6 and 7 for tariffs.

As was to be expected, increases in iceberg trade costs and tariffs have approximately the same effects for Britain, but they have very different effects for other EU countries. Focusing on Britain and tariffs, in the benchmark economy, the overall change in welfare is approximately -0.6%. This welfare reduction is distributed unequally across the different factors of production. The changes in real incomes of capital, low-, medium-, and high-skilled labor are $(-0.9\%, -0.1\%, -0.3\%, -0.7\%)$. All factors of production lose, but the losses are largest for capital and high-skilled workers.

In the HVA economy, the overall change is approximately the same at -0.6%. This was to be expected given that the increases in tariffs act on Britain approximately like increases in iceberg trade costs, and given that the HVA economy matches the volume of trade of the benchmark economy and than the data. However, the distribution of this welfare reduction across factors displays some significant differences. The changes in real incomes of capital, low-, medium-, and high-skilled labor are $(-1.3\%, -0.1\%, -0.2\%, -0.6\%)$.

In the LVA model, the overall change in welfare is approximately half at -0.3%. This was to be expected given that the LVA economy features a lower volume of trade than the benchmark economy and than the data. The changes in the real incomes of capital, low-, medium-, and high-skilled labor are $(-0.9\%, 0.4\%, 0.1\%, -0.1\%)$. The real income of low and medium-skilled labor actually increases. Basically, the price of capital and the wage of high-skilled labor falls by enough to offset the reduction in the wage of the low and medium skilled workers.

Overall, we see significant differences between the benchmark economy and the HVA economy, and even bigger ones between the benchmark economy and the LVA economy. Accounting for intermediate goods and input-output linkages significantly changes the picture of Brexit.

Scenario III: US Tariffs on China

The last scenario illustrates how tariffs between two countries can affect a third country via supply chains. The Trump administration has recently imposed a 10% tariff on a broad range of Chinese imports, with the possibility that these tariffs will be increased further to 25%. Meanwhile, the administration renegotiated a free-trade deal with Mexico. In this subsection, we analyze an impressionistic version of this policy, computing the effects of the imposition of a tariff on China while maintaining free trade with Mexico.

We stress at the outset that our results should be interpreted as long-run effects, once prices have adjusted and factors and inputs have been reallocated. This long-run focus explains the difference between our findings and those of recent studies (Fajgelbaum et al., 2019; Amiti et al., 2019), who investigate the effects of tariffs in the short run over a horizon of less than a year.

In Table 3, we show the effects on the welfare of a few countries of a 10% tax on Chinese imports in the benchmark economy and in the LVA economy. We start with the benchmark economy. As one might expect, China is affected most negatively by this policy with a change in welfare of -0.24%, followed by some of China's close trading

	AUS	BRA	CAN	CHN	KOR	MEX	TWN	USA	World
Benchmark	-0.01%	0.00%	0.02%	-0.24%	-0.02%	0.03%	-0.06%	0.01%	-0.01%
LVA	-0.01%	0.00%	0.01%	-0.04%	0.00%	0.01%	-0.01%	0.00%	0.00%
HVA	-0.00%	0.00%	0.02%	-0.18%	0.02%	0.03%	0.02%	0.00%	0.00%

Table 3: The change in welfare for a subset of countries when the USA imposes a 10% import tariff on all Chinese trade. We compare the results from the benchmark economy with intermediate goods and input-output linkages with an economy that assumes only value-added production functions (LVA).

partners, like Taiwan, Korea, and Australia. The United States gains very slightly by 0.01% from the imposition of tariffs, illustrating almost fully offsetting effects on tariff revenues 0.12% and real factor income -0.11%. Interestingly, the country that gains the most from the tariffs is Mexico at 0.03% due to the resulting increase in the demand for Mexican goods.

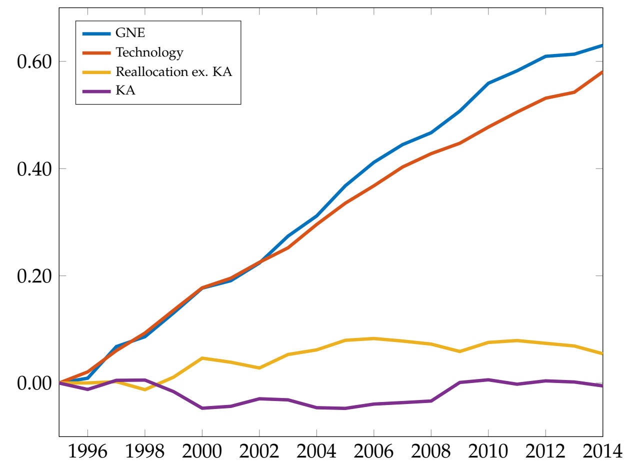
Table 8 in Appendix B breaks down how the different factor prices change in response to the tariffs. This table shows that the overall trend, across most countries including the USA and Mexico but excluding China, is that capital loses and labor gains. For most countries except China and Mexico, most of labor's gains accrue to high-skilled workers. For example, in the USA, capital loses by -0.36%, low-skilled labor gains by 0.08%, medium skilled labor gains by 0.03%, and high-skilled labor gains by 0.13%. For Mexico, most of the gains go to low- and medium-skilled labor instead, by 0.41% and 0.63% respectively. In China, all factors lose but low-skilled labor and high-skilled labor actually lose the most, by -0.17% and -0.44% respectively.

In the LVA economy, the welfare effects are much smaller overall. The most impressive difference is for China, which now loses by only -0.04% compared to -0.24% in the benchmark economy. Once again, taking into account input-output linkages matters for assessing Trump tariffs.

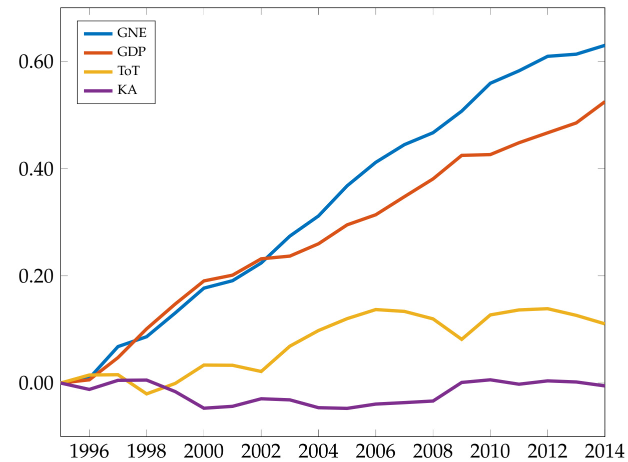
7.2 Welfare-Accounting

We end our applications by decomposing the change in real expenditure in different countries over time. We implement our two decompositions: the reallocation decomposition and the terms-of-trade decomposition. We abstract away from distortions. Unlike our previous applications, these decompositions are non-parametric in the sense that they do not require taking a stand on the various elasticities of substitution.

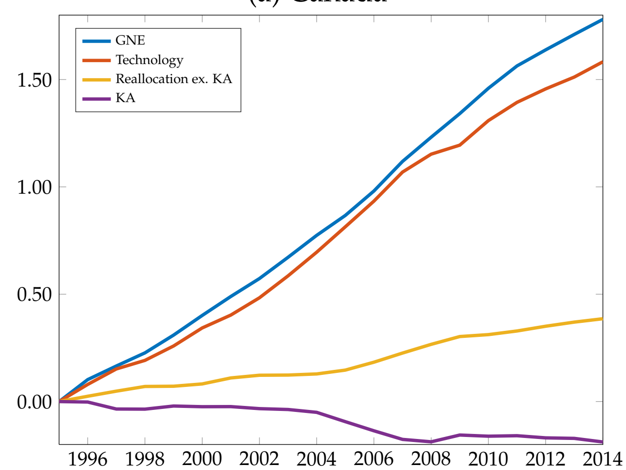
The left column of Figure 4 displays the cumulative change in each component over



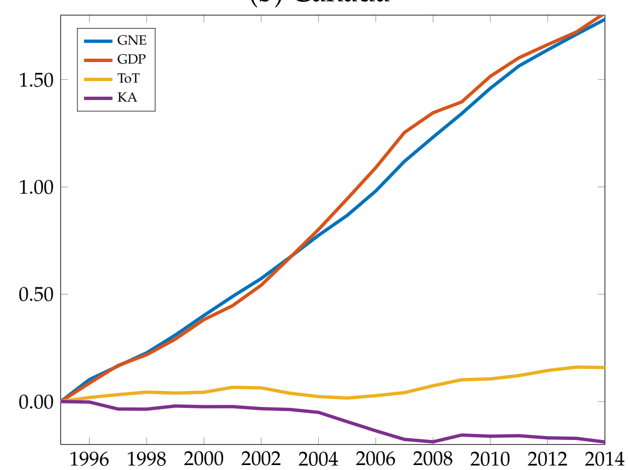
(a) Canada



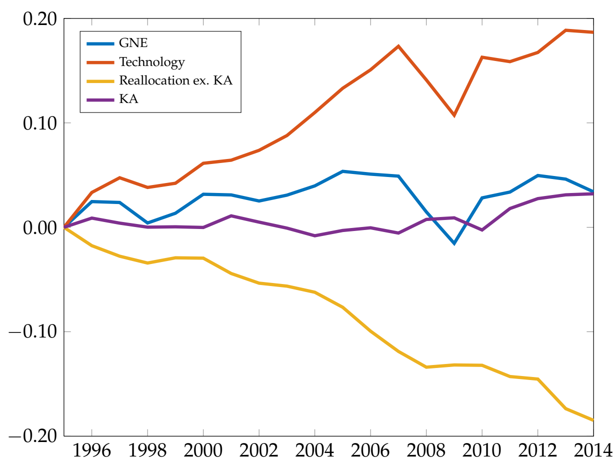
(b) Canada



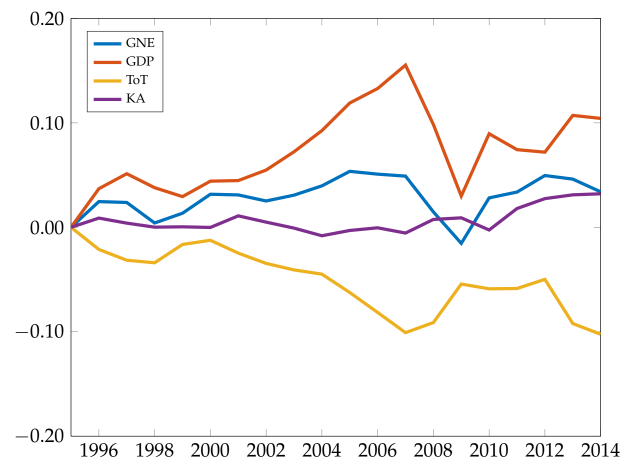
(c) China



(d) China



(e) Japan



(f) Japan

Figure 4: Welfare accounting according to the reallocation decomposition (left column) and according to the terms-of-trade decomposition (right column), for a sample of countries, using the WIOD data.

time of the reallocation decomposition, for a few countries (Canada, China, and Japan). We choose these three countries because they depict a systematic pattern: industrializing countries, like China, and commodities- or services-dependent industrialized countries, like Canada, are experiencing positive reallocation, whereas manufacturing-dependent industrialized countries, like Japan, are experiencing negative reallocation.

The right column of Figure 4 displays the terms-of-trade decomposition. Commodity producers like Canada experience large movements in terms of trade due to fluctuations in commodity prices. Even for countries for which terms-of-trade effects are small, reallocation effects are typically large, indicating that these countries cannot be taken to be approximately closed.

Finally, it is interesting to note that the difference between the reallocation effect on the one hand, and the terms of trade effect and the transfer effect on the other hand identifies the following technological residual:[42]

$$\sum_{i\in N}((\chi_c^Y/\chi_c^W)\,\lambda_i^{Y_c}-\lambda_i^{W_c})\,\mathrm{d}\log A_i+\sum_{f\in F}((\chi_c^Y/\chi_c^W)\Lambda_f^{Y_c}-\Lambda_f^{W_c})\,\mathrm{d}\log L_c.$$

This residual is a measure of the difference between country c's technological change and its exposure to world technical change, including the effects of changes in productivities and in factor supplies. For a closed economy, it is always zero. By comparing the two columns of Figure 4, we can see that (and by how much) China and Canada are experiencing faster growth in productivities and factor supplies in their domestic real output than in their consumption baskets, while the pattern is reversed for Japan.

8 Conclusion

In this paper, we establish a unified framework for studying output and income in efficient and distorted open-economies. We provide ex-post sufficient statistics for measurement and ex-ante sufficient statistics for conducting counterfactuals. Our formulas provide new characterizations of the gains from trade, the losses from trade protectionism, the aggregation of trade elasticities, and the distributional consequences of trade policy.

[42]That is, we compute $(\partial\log\mathcal{W}_h)(\partial\mathcal{X})\,\mathrm{d}\,\mathcal{X}-(1/\chi_c^W)\,\mathrm{d}\,T_c-(\chi_c^Y/\chi_c^W)\,\mathrm{d}\log P_{Y_c}+\mathrm{d}\log P_{W_c}$.

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A Data Appendix

To conduct the counterfactual exercises in Section 7.1, we use the World Input-Output Database Timmer et al. (2015). We use the 2013 release of the data for the final year which has no-missing data — that is 2008. We use the 2013 release because it has more detailed information on the factor usage by industry. We aggregate the 35 industries in the database to get 30 industries to eliminate missing values, and zero domestic production shares, from the data. In Table 4, we list our aggregation scheme, as well as the elasticity of substitution, based on Caliendo and Parro (2015) and taken from Costinot and Rodriguez-Clare (2014) associated with each industry. We calibrate the model to match the input-output tables and the socio-economic accounts tables in terms of expenditure shares in steady-state (before the shock).

For the growth accounting exercise in Section 7.1, we use both the 2013 and the 2016 release of the WIOD data. When we combine this data, we are able to cover a larger number of years. We compute our growth accounting decompositions for each release of the data separately, and then paste the resulting decompositions together starting with the year of overlap. To construct the consumer price index and the GDP deflator for each country, we use the final consumption weights or GDP weights of each country in each year to sum up the log price changes of each good. To arrive at the price of each good, we use the gross output prices from the socio-economic accounts tables which are reported at the (country of origin, industry) level into US dollars using the contemporaneous exchange rate, and then take log differences. This means that we assume that the log-change in the price of each good at the (origin, destination, industry of supply, industry of use) level is the same as (origin, industry of supply) level. If there are differential (changing) transportation costs over time, then this assumption is violated.

To arrive at the contemporaneous exchange rate, we use the measures of nominal GDP in the socio-economic accounts for each year (reported in local currency) to nominal GDP in the world input-output database (reported in US dollars).

B Additional Results From the Quantitative Model

Tables 5, 6, 7, 8 tabulate more results from the scenarios in Section 7.1.

	WIOD Sector	Aggregated sector	Trade Elasticity
1	Agriculture, Hunting, Forestry and Fishing	1	8.11
2	Mining and Quarrying	2	15.72
3	Food, Beverages and Tobacco	3	2.55
4	Textiles and Textile Products	4	5.56
5	Leather, Leather and Footwear	4	5.56
6	Wood and Products of Wood and Cork	5	10.83
7	Pulp, Paper, Paper , Printing and Publishing	6	9.07
8	Coke, Refined Petroleum and Nuclear Fuel	7	51.08
9	Chemicals and Chemical Products	8	4.75
10	Rubber and Plastics	8	4.75
11	Other Non-Metallic Mineral	9	2.76
12	Basic Metals and Fabricated Metal	10	7.99
13	Machinery, Nec	11	1.52
14	Electrical and Optical Equipment	12	10.6
15	Transport Equipment	13	0.37
16	Manufacturing, Nec; Recycling	14	5
17	Electricity, Gas and Water Supply	15	5
18	Construction	16	5
19	Sale, Maintenance and Repair of Motor Vehicles...	17	5
20	Wholesale Trade and Commission Trade, ...	17	5
21	Retail Trade, Except of Motor Vehicles and...	18	5
22	Hotels and Restaurants	19	5
23	Inland Transport	20	5
24	Water Transport	21	5
25	Air Transport	22	5
26	Other Supporting and Auxiliary Transport....	23	5
27	Post and Telecommunications	24	5
28	Financial Intermediation	25	5
29	Real Estate Activities	26	5
30	Renting of M&Eq and Other Business Activities	27	5
31	Public Admin/Defence; Compulsory Social Security	28	5
32	Education	29	5
33	Health and Social Work	30	5
34	Other Community, Social and Personal Services	30	5
35	Private Households with Employed Persons	30	5

Table 4: The sectors in the 2013 release of the WIOD data, and the aggregated sectors in our data.

	Benchmark Iceberg					LVA Iceberg				
	Capital	Low	Medium	High	Welfare	Capital	Low	Medium	High	Welfare
AUS	0.06%	-0.02%	-0.02%	-0.04%	0.01%	0.05%	-0.02%	-0.03%	-0.02%	0.01%
AUT	-0.07%	-0.11%	-0.06%	0.00%	-0.05%	-0.08%	-0.06%	-0.02%	0.04%	-0.03%
BEL	-0.26%	-0.35%	-0.22%	0.03%	-0.20%	-0.15%	-0.23%	-0.11%	0.11%	-0.11%
BGR	-0.13%	-0.26%	-0.03%	0.06%	-0.16%	-0.08%	-0.13%	0.03%	0.08%	-0.08%
BRA	0.02%	-0.01%	-0.01%	0.00%	0.00%	0.01%	-0.01%	-0.02%	0.01%	0.00%
CAN	0.02%	0.07%	0.02%	-0.03%	0.01%	0.02%	0.06%	0.01%	-0.02%	0.00%
CHN	0.01%	0.00%	0.00%	-0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%
CYP	-0.32%	-0.51%	-0.18%	-0.10%	-0.28%	-0.20%	-0.38%	-0.04%	0.06%	-0.12%
CZE	-0.06%	-0.18%	-0.15%	0.02%	-0.08%	-0.08%	-0.09%	-0.07%	0.09%	-0.06%
DEU	-0.18%	-0.21%	-0.09%	-0.07%	-0.11%	-0.11%	-0.15%	-0.02%	-0.02%	-0.05%
DNK	-0.14%	-0.32%	-0.17%	0.00%	-0.13%	-0.10%	-0.21%	-0.10%	0.08%	-0.07%
ESP	-0.04%	-0.06%	-0.05%	-0.07%	-0.07%	-0.02%	-0.03%	-0.01%	-0.02%	-0.03%
EST	-0.17%	-0.16%	-0.14%	-0.02%	-0.13%	-0.15%	-0.07%	-0.04%	0.07%	-0.06%
FIN	-0.26%	-0.14%	-0.14%	0.05%	-0.11%	-0.18%	-0.08%	-0.08%	0.08%	-0.07%
FRA	-0.10%	-0.08%	-0.08%	-0.02%	-0.07%	-0.05%	-0.04%	-0.04%	0.01%	-0.04%
GBR	-0.96%	-0.10%	-0.28%	-0.52%	-0.54%	-0.82%	0.33%	0.13%	-0.12%	-0.27%
GRC	-0.09%	0.02%	0.01%	-0.07%	-0.07%	-0.06%	0.04%	0.03%	-0.03%	-0.03%
HUN	-0.11%	-0.18%	-0.16%	0.05%	-0.09%	-0.12%	-0.11%	-0.09%	0.10%	-0.06%
IDN	0.01%	-0.01%	0.01%	-0.05%	0.00%	0.01%	-0.02%	0.00%	-0.01%	0.00%
IND	0.02%	-0.03%	-0.02%	0.05%	0.00%	0.00%	-0.02%	-0.01%	0.07%	0.00%
IRL	-1.97%	-1.20%	-1.02%	-0.72%	-1.42%	-0.97%	-0.59%	-0.62%	-0.30%	-0.66%
ITA	-0.03%	-0.10%	-0.03%	0.00%	-0.04%	-0.04%	-0.05%	0.00%	0.03%	-0.02%
JPN	0.00%	0.02%	0.01%	-0.01%	0.00%	0.00%	0.02%	0.01%	0.00%	0.00%
KOR	0.00%	0.01%	0.03%	0.00%	0.01%	-0.01%	0.00%	0.01%	0.00%	0.00%
LTU	-0.04%	-0.11%	-0.10%	-0.04%	-0.08%	-0.07%	-0.02%	-0.01%	0.03%	-0.03%
LUX	-1.64%	1.00%	-0.31%	-0.54%	-1.32%	-0.82%	-1.44%	-1.13%	-0.89%	-1.04%
LVA	-0.06%	-0.17%	-0.12%	-0.01%	-0.09%	-0.04%	-0.08%	-0.07%	0.02%	-0.04%
MEX	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%	-0.01%	0.00%
MLT	-0.63%	-1.01%	-0.61%	-0.38%	-0.69%	-0.47%	-0.54%	-0.14%	0.05%	-0.36%
NLD	-0.19%	-0.52%	-0.36%	0.01%	-0.23%	-0.10%	-0.31%	-0.19%	0.06%	-0.10%
POL	-0.01%	-0.24%	-0.20%	0.01%	-0.08%	0.01%	-0.14%	-0.15%	0.04%	-0.04%
PRT	0.04%	-0.21%	-0.03%	0.01%	-0.09%	-0.02%	-0.11%	0.02%	0.05%	-0.04%
ROU	-0.03%	-0.11%	-0.07%	-0.05%	-0.09%	0.00%	-0.08%	-0.05%	0.00%	-0.03%
RUS	-0.01%	0.00%	0.04%	0.02%	0.03%	-0.04%	-0.02%	0.04%	0.03%	0.02%
SVK	-0.05%	-0.14%	-0.08%	0.02%	-0.05%	-0.05%	-0.09%	-0.03%	0.10%	-0.04%
SVN	-0.07%	-0.13%	-0.08%	0.00%	-0.07%	-0.07%	-0.09%	-0.03%	0.04%	-0.03%
SWE	-0.23%	-0.22%	-0.16%	-0.04%	-0.16%	-0.13%	-0.12%	-0.08%	0.01%	-0.08%
TUR	0.01%	0.01%	0.01%	-0.01%	0.00%	0.01%	-0.01%	-0.01%	0.01%	0.00%
TWN	0.04%	0.01%	-0.01%	-0.02%	0.02%	0.01%	0.00%	-0.01%	-0.01%	0.00%
USA	0.00%	0.02%	0.01%	-0.01%	0.00%	-0.01%	0.03%	0.02%	0.00%	0.00%
ROW	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%

Table 5: The log change in real wages of each factor, in terms of the domestic real expenditure delfator, as well as the log change in welfare for each country as a result of an increase in iceberg transport costs of British imports into the EU of 10%. The table shows results for both the Benchmark economy and the LVA economy.

	Benchmark Tariff					LVA Tariff				
	Capital	Low	Medium	High	Welfare	Capital	Low	Medium	High	Welfare
AUS	0.07%	-0.02%	-0.02%	-0.05%	0.01%	0.05%	-0.02%	-0.03%	-0.02%	0.01%
AUT	-0.08%	-0.11%	-0.06%	0.03%	0.05%	-0.09%	-0.06%	-0.02%	0.07%	0.02%
BEL	-0.25%	-0.37%	-0.20%	0.08%	0.07%	-0.16%	-0.26%	-0.10%	0.17%	0.01%
BGR	-0.13%	-0.25%	0.02%	0.16%	0.00%	-0.09%	-0.13%	0.06%	0.15%	-0.01%
BRA	0.01%	-0.01%	-0.01%	0.00%	0.00%	0.01%	-0.01%	-0.02%	0.01%	0.00%
CAN	0.02%	0.08%	0.03%	-0.04%	0.01%	0.02%	0.07%	0.01%	-0.03%	0.00%
CHN	0.01%	0.00%	0.00%	-0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%
CYP	-0.28%	-0.61%	-0.15%	0.08%	0.02%	-0.18%	-0.47%	-0.05%	0.14%	0.02%
CZE	-0.05%	-0.22%	-0.15%	0.07%	0.08%	-0.07%	-0.12%	-0.07%	0.12%	0.01%
DEU	-0.18%	-0.22%	-0.09%	-0.04%	0.04%	-0.11%	-0.16%	-0.02%	0.00%	0.02%
DNK	-0.21%	-0.32%	-0.15%	0.09%	0.06%	-0.13%	-0.22%	-0.09%	0.12%	0.02%
ESP	-0.04%	-0.06%	-0.05%	-0.04%	0.00%	-0.02%	-0.04%	-0.01%	0.00%	0.00%
EST	-0.20%	-0.15%	-0.13%	0.06%	0.05%	-0.17%	-0.09%	-0.04%	0.12%	0.02%
FIN	-0.29%	-0.13%	-0.11%	0.10%	0.07%	-0.20%	-0.06%	-0.07%	0.12%	0.03%
FRA	-0.08%	-0.08%	-0.08%	-0.02%	0.01%	-0.05%	-0.04%	-0.04%	0.02%	0.00%
GBR	-0.93%	-0.11%	-0.35%	-0.70%	-0.60%	-0.87%	0.40%	0.14%	-0.19%	-0.30%
GRC	-0.10%	0.03%	0.03%	-0.03%	-0.02%	-0.07%	0.04%	0.04%	-0.01%	0.00%
HUN	-0.15%	-0.17%	-0.14%	0.11%	0.06%	-0.14%	-0.12%	-0.08%	0.15%	0.01%
IDN	0.01%	-0.01%	0.01%	-0.04%	0.00%	0.01%	-0.02%	0.00%	-0.01%	0.00%
IND	0.01%	-0.04%	-0.02%	0.06%	0.00%	-0.01%	-0.02%	-0.01%	0.09%	0.00%
IRL	-2.63%	-0.61%	-0.43%	-0.02%	0.51%	-1.26%	-0.36%	-0.37%	0.05%	0.23%
ITA	-0.02%	-0.11%	-0.03%	0.03%	0.02%	-0.04%	-0.06%	0.00%	0.05%	0.00%
JPN	0.00%	0.02%	0.01%	-0.01%	0.00%	-0.01%	0.02%	0.01%	0.00%	0.00%
KOR	-0.01%	0.01%	0.04%	0.00%	0.01%	-0.01%	0.00%	0.02%	0.00%	0.00%
LTU	-0.05%	-0.11%	-0.09%	-0.01%	-0.01%	-0.08%	-0.02%	0.00%	0.05%	-0.01%
LUX	-1.61%	2.27%	0.41%	-0.47%	3.15%	-0.71%	-1.06%	-0.97%	-0.78%	0.33%
LVA	-0.07%	-0.17%	-0.12%	0.02%	-0.02%	-0.04%	-0.09%	-0.07%	0.04%	0.00%
MEX	0.00%	0.01%	0.01%	0.00%	0.00%	0.00%	0.02%	0.01%	-0.02%	0.00%
MLT	-0.73%	-0.85%	-0.42%	-0.05%	0.16%	-0.53%	-0.53%	-0.03%	0.25%	0.07%
NLD	-0.24%	-0.50%	-0.29%	0.07%	0.05%	-0.13%	-0.31%	-0.16%	0.11%	0.04%
POL	-0.01%	-0.24%	-0.22%	0.08%	0.05%	0.02%	-0.15%	-0.17%	0.08%	0.02%
PRT	0.05%	-0.21%	-0.02%	0.07%	0.02%	-0.02%	-0.12%	0.04%	0.09%	0.01%
ROU	-0.01%	-0.12%	-0.08%	-0.03%	-0.01%	0.02%	-0.09%	-0.06%	0.01%	0.00%
RUS	-0.01%	0.00%	0.03%	0.01%	0.02%	-0.04%	-0.02%	0.05%	0.03%	0.02%
SVK	-0.04%	-0.14%	-0.08%	0.07%	0.07%	-0.05%	-0.09%	-0.03%	0.14%	0.02%
SVN	-0.08%	-0.15%	-0.08%	0.04%	0.02%	-0.08%	-0.11%	-0.03%	0.06%	0.01%
SWE	-0.27%	-0.23%	-0.14%	0.05%	0.05%	-0.16%	-0.12%	-0.07%	0.06%	0.02%
TUR	0.01%	0.02%	0.01%	-0.02%	0.00%	0.01%	-0.01%	-0.01%	0.01%	0.01%
TWN	0.04%	0.01%	-0.01%	-0.02%	0.02%	0.01%	0.00%	-0.01%	-0.01%	0.00%
USA	-0.01%	0.03%	0.02%	0.00%	0.00%	-0.02%	0.04%	0.02%	0.01%	0.00%
ROW	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%

Table 6: The percentage change in the real wages of each factor, in terms of the real expenditure deflator, as well as the percentage in welfare or each country as a result of an increase in tariffs on British imports into EU-member countries of 10%. The table shows results for both the Benchmark and LVA economies.

	Overall	Factors	Transfers	Tariff Revenue
AUT	0.05%	-0.06%	0.01%	0.09%
BEL	0.07%	-0.19%	0.01%	0.25%
BGR	0.00%	-0.10%	-0.05%	0.14%
CYP	0.02%	-0.13%	-0.12%	0.27%
CZE	0.07%	-0.09%	0.02%	0.14%
DEU	0.04%	-0.13%	0.02%	0.15%
DNK	0.06%	-0.13%	0.02%	0.18%
ESP	0.00%	-0.04%	-0.02%	0.06%
EST	0.05%	-0.10%	-0.01%	0.16%
FIN	0.07%	-0.12%	0.02%	0.17%
FRA	0.01%	-0.06%	0.00%	0.08%
GBR	-0.58%	-0.59%	0.01%	0.00%
GRC	-0.02%	-0.03%	-0.03%	0.05%
HUN	0.05%	-0.09%	0.00%	0.14%
IRL	0.49%	-1.36%	0.13%	1.72%
ITA	0.02%	-0.04%	0.00%	0.06%
LTU	-0.01%	-0.04%	-0.02%	0.06%
LUX	3.12%	-0.68%	0.01%	3.80%
LVA	-0.01%	-0.05%	-0.03%	0.07%
NLD	0.15%	-0.55%	-0.06%	0.76%
NOR	0.05%	-0.23%	0.02%	0.26%
POL	0.02%	-0.04%	-0.04%	0.09%
SVK	0.07%	-0.04%	0.00%	0.11%
SVN	0.01%	-0.06%	-0.01%	0.08%
SWE	0.05%	-0.16%	0.02%	0.19%

Table 7: The change in welfare for each EU country, as well as the contribution of each component of real income (factor returns, transfers, and tariff revenue) to this overall change from Brexit.

	AUS	AUT	BEL	BGR	BRA	CAN	CHN	CYP	CZE	DEU
Capital	-0.11%	0.01%	-0.01%	0.02%	-0.04%	-0.27%	-0.14%	0.02%	0.00%	0.00%
Low	0.07%	0.00%	-0.01%	0.00%	0.01%	0.30%	-0.17%	0.02%	0.01%	0.01%
Med	0.06%	0.01%	0.00%	0.02%	0.03%	0.20%	-0.08%	0.04%	0.02%	0.01%
High	0.05%	0.00%	0.03%	0.04%	0.02%	0.25%	-0.44%	0.05%	0.05%	0.02%

	DNK	ESP	EST	FIN	FRA	GBR	GRC	HUN	IDN	IND
Capital	-0.03%	0.00%	0.04%	0.02%	0.01%	-0.01%	-0.02%	-0.01%	-0.03%	-0.04%
Low	0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.02%	0.02%	0.05%	0.06%
Med	0.02%	0.01%	0.00%	0.00%	0.00%	0.01%	0.02%	0.02%	0.06%	0.07%
High	0.03%	0.03%	0.02%	0.01%	0.01%	0.03%	0.02%	0.04%	0.05%	0.08%

	IRL	ITA	JPN	KOR	LTU	LUX	LVA	MEX	NLD	NOR
Capital	-0.01%	0.00%	-0.01%	-0.11%	0.04%	0.02%	0.01%	-0.18%	0.04%	0.01%
Low	-0.05%	0.00%	0.00%	0.02%	-0.01%	-0.06%	0.00%	0.41%	-0.01%	0.01%
Med	-0.01%	0.01%	0.01%	0.01%	-0.02%	-0.02%	0.00%	0.63%	0.00%	0.01%
High	0.04%	0.02%	0.02%	0.03%	0.00%	0.02%	0.01%	0.10%	0.04%	0.02%

	POL	PRT	ROU	RUS	SVK	SVN	SWE	TUR	TWN	USA
Capital	0.01%	0.01%	0.01%	0.02%	0.01%	0.00%	0.00%	0.00%	-0.15%	-0.36%
Low	0.00%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.02%	-0.01%	0.08%
Med	0.00%	0.02%	0.00%	0.00%	0.01%	0.00%	0.01%	0.00%	0.02%	0.03%
High	0.02%	0.04%	0.02%	0.01%	0.04%	0.02%	0.01%	0.01%	0.07%	0.13%

Table 8: The change in factor rewards, in units of domestic consumption, for each primary factor in response to a 10% increase in USA taxes on Chinese imports.

C Beyond CES

Following Baqaee and Farhi (2017a), we show how all the results in this paper can be easily generalized to arbitrary neoclassical production functions simply by replacing the input-output covariance operator with the *input-output substitution operator* instead.

For a producer k with cost function $\mathbf{C}_k$, the Allen-Uzawa elasticity of substitution between inputs x and y is

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2\mathbf{C}_k/(dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and Ω_{ky} is the expenditure share in cost of input y. We also use this definition for final demand aggregators.

The *input-output substitution operator* for producer k is defined as

$$\begin{aligned}\Phi_k(\Psi_{(i)}, \Psi_{(j)}) &= -\sum_{x,y \in N+F} \Omega_{kx}[\delta_{xy} + \Omega_{ky}(\theta_k(x,y) - 1)]\Psi_{xi}\Psi_{yj}, \\ &= \frac{1}{2} E_{\Omega^{(k)}}\left((\theta_k(x,y) - 1)(\Psi_i(x) - \Psi_i(y))(\Psi_j(x) - \Psi_j(y))\right),\end{aligned}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y.

In the CES case with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\theta_k(x,y) = \theta_k$ if $x \neq y$, and the own Allen-Uzawa elasticities are given by $\theta_k(x,x) = -\theta_k(1 - \Omega_{kx})/\Omega_{kx}$. It is easy to verify that we then recover the input-output covariance operator:

$$\Phi_k(\Psi_{(i)}, \Psi_{(j)}) = (\theta_k - 1)Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)}).$$

Even outside the CES case, the input-output substitution operator shares many properties with the input-output covariance operator. For example, it is immediate to verify, that: $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$ is bilinear in $\Psi_{(i)}$ and $\Psi_{(j)}$; $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$ is symmetric in $\Psi_{(i)}$ and $\Psi_{(j)}$; and $\Phi_k(\Psi_{(i)}, \Psi_{(j)}) = 0$ whenever $\Psi_{(i)}$ or $\Psi_{(j)}$ is a constant.

All the structural results in the paper can be extended to general non-CES economies by simply replacing terms of the form $(\theta_k - 1)Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)})$ by $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$.

D Duality with Multiple Factors

The duality between trade shocks in an open economy and productivity shocks in a closed economy extends beyond the one-factor case. In the multi-factor case, trade shocks in an open economy translate into productivity shocks *and* shocks to factor prices in the closed economy. In this section, we establish this duality. As an example application, in Section H, we show how the model in Galle et al. (2017), which studies the distributional consequences of trade with a Roy model, can be generalized to economies with production networks.

With multiple factors, we must use the change in the dual price deflator $\Delta \log \check{P}_{W_c} = \Delta \log \check{P}_{Y_c} = \Delta \log \check{p}_c$ of the dual economy for given changes in factor prices and not the change in real expenditure or welfare for given factor supplies. This requires the choice of a numeraire in the dual closed economy: we use the nominal GDP, which means that we normalize the nominal GDP of the dual closed economy to one.

Theorem 8 (Exact Duality). *The discrete change in welfare $\Delta \log W_c$ of the original open economy in response to discrete shocks to iceberg trade costs or productivities outside of country c is equal to (minus) the discrete change in the price deflator $-\Delta \log \check{P}_{Y_c}$ of the dual closed economy in response to discrete shocks to productivities $\Delta \log \check{A}_i = -(1/\varepsilon_i)\Delta \log \Omega_{ic}$ and discrete shocks to factor wages $\Delta \log \check{A}_f = -\Delta \log \Lambda_f^c$. This duality result is global in that it holds exactly for arbitrarily large shocks.*

Corollary 8 (First-Order Duality). *A first-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i,$$

where applying Hulten's theorem, $\check{\lambda}_i$ is the sales share of producer i when $\in M_c$ and the sales share of factor i in the dual closed economy (which we also sometimes write $\check{\Lambda}_i$).

Corollary 9 (Second-Order Duality). *A second-order approximation to the change in welfare of the original open economy is:*

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i - \frac{1}{2} \sum_{i,j \in M_c + F_c} \frac{d^2 \log \check{P}_{Y_c}}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log A_i} \Delta \log \check{A}_j \Delta \log \check{A}_i,$$

where applying Baqaee and Farhi (2017a),

$$-\frac{d^2 \log \check{P}_{Y_c}}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i} = \frac{\mathrm{d}\, \check{\lambda}_i}{\mathrm{d} \log \check{A}_j} = \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Cov_{\check{\Omega}^{(k)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right).$$

We can re-express the second-order approximation to the change in welfare of the original open economy as:

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} \approx \sum_{i \in M_c + F_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2} \sum_{k \in N_c} (\theta_k - 1) \check{\lambda}_k Var_{\check{\Omega}^{(k)}} \left(\sum_{i \in M_c + F_c} \check{\Psi}_{(i)} \Delta \log \check{A}_i \right).$$

Corollary 10 (Exact Duality and Nonlinearities with an Industry Structure)**.** *For country c with an industry structure, we have the following exact characterization of the nonlinearities in welfare changes of the original open economy.*

(i) (Industry Elasticities) Consider two economies with the same initial input-output matrix and industry structure, the same trade elasticities, but with lower elasticities across industries for one than for the other so that $\theta_\kappa \leq \theta'_\kappa$ for all industries κ. Then $\Delta \log W_c = \Delta \log \check{Y}_c \leq \Delta \log W'_c = \Delta \log \check{Y}'_c$ so that negative (positive) shocks have larger negative (smaller positive) welfare effects in the economy with the lower industry elasticities.

(ii) (Cobb-Douglas) Suppose that all the elasticities of substitution across industries (and with the factor) are equal to unity ($\theta_\kappa = 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is linear in $\Delta \log \check{A}$.

(iii) (Complementarities) Suppose that all the elasticities of substitution across industries (and with the factor) are below unity ($\theta_\kappa \leq 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is concave in $\Delta \log \check{A}$, and so nonlinearities amplify negative shocks and mitigate positive shocks.

(iv) (Substituabilities) Suppose that all the elasticities of substitution across industries (and with the factor) are above unity ($\theta_\kappa \geq 1$), then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is convex in $\Delta \log \check{A}$, and so nonlinearities mitigate negative shocks and amplify positive shocks.

(v) (Exposure Heterogeneities) Suppose that industry κ is uniformly exposed to the shocks as they unfold, so that $Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) = 0$ for all s where s indexes the dual closed economy with productivity shocks $\Delta \log \check{A}_{\iota,s} = s \Delta \log \check{A}_\iota$, then $\Delta \log W_c = -\Delta \log \check{P}_{Y_c}$ is independent of θ_κ. Furthermore

$$\Delta \log W_c = -\Delta \log \check{P}_{Y_c} = \sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\lambda}_\iota \Delta \log \check{A}_\iota + \int_0^1 \sum_{\kappa \in \mathcal{N}_c} (\theta_\kappa - 1) \check{\lambda}_{\kappa,s} Var_{\check{\Omega}_s^{(\kappa)}} \left(\sum_{\iota \in \mathcal{M}_c + \mathcal{F}_c} \check{\Psi}_{(\iota),s} \Delta \log \check{A}_\iota \right) (1-s) ds.$$

E Generalizing Sections 3 and 5 with Distortions

In this section, we explain how to adapt the results of Sections 3 and 5 in economies with tariffs or other distortions.

Comparative Statics: Ex-Ante Sufficient Statistics

Since any wedge can be represented as a markup, without loss of generality, we assume that all wedges in the economy (including tariffs) have been represented as markups.[43] We use the diagonal matrix of markups/wedges μ. Following Baqaee and Farhi (2017b), we define the *cost-based* HAIO matrix $\tilde{\Omega} = \mu\Omega$ and the corresponding cost-based Leontief inverse matrix $\tilde{\Psi} = (I - \tilde{\Omega})^{-1}$. All the exposures and factor income shares that we defined with the matrix Ω have cost-based analogues which we denote with tildes.

Finally, it is convenient to introduce "fictitious" factors, one for each producer $i \in N$, which collects the revenues $\lambda_i(1 - 1/\mu_i)$ earned by the markup/wedge of this producer. We denote the set of true and fictitious factors to be F^*. For each fictitious factor $f \in F^* - F$, we denote by $\iota(f) \in N$ the good associated with it. Just like for a true factor, we define Φ_{cf} for a fictitious factor to be the share of the income of this factor which accrues to the representative agent of country c. All exposures in gross real output, and in real expenditure or welfare to a fictitious factor are equal to zero, at the country and world levels. But the incomes shares of these factors are not zero. For example $\tilde{\Lambda}_f^{W_c} = \tilde{\Lambda}_f^{W} = 0$, but $\Lambda_f^c \neq 0$ and $\Lambda_f \neq 0$ if the markup/wedge of the corresponding producer is nonzero.

We can characterize changes in real output $d\log Y_c = \sum_{i\in N} \chi_i^{Y_c} d\log q_{ci}$ exactly as in the model without distortions, where recall that $q_{ci} \geq 0$ for $i \in N_c$ and $q_{ci} \leq 0$ for $i \notin N_c$. However, this notion is less interesting in the presence of distortions, for example to compute changes in aggregate country productivity, because the double-deflation method runs into conceptual problems. Instead, we define the change in the *gross* real output $d\log \hat{Y}_c = \sum_{i\in N_c} \chi_i^{\hat{Y}_c} d\log q_{ci}$ of a country by treating imports in the same way as factor inputs, where $\chi_i^{\hat{Y}_c} = p_i q_{c_i}/(\sum_{i\in N_c} p_i q_{ci})$. Following the by now usual template, we also define the corresponding revenue- and cost-based exposures to goods or factors k as $\lambda_k^{\hat{Y}_c}$ and $\tilde{\lambda}_k^{\hat{Y}_c}$.

We now state two growth-accounting theorems, one for changes in gross real output and the other for changes in real expenditure or welfare at the country and world levels. These theorems offer decompositions into "pure" technology effects and reallocation effects. As for the case without distortions discussed in the main text, we slightly abuse

[43] To represent a wedge on i's ability purchase inputs from k, we can introduce a new producer which buys from k and sells to i at a markup.

notation: first, except for changes in welfare at the country level, these objects are not differentials of corresponding level functions; second except for changes in welfare at the country level, reallocation effects are defined as the the changes in the corresponding object with fixed prices (not chained) and holding the allocation matrix constant (this is not necessary for the change in welfare at the country level because it is the differential of a function, which can be evaluated with a constant allocation matrix, along the lines of the exposition in the main text).

Theorem 9 (Output-Accounting)**.** *The change in gross real output of country c to productivity shocks, factor supply shocks, transfer shocks, and shocks to markups/wedges, can be decomposed into the "pure" effects of changes in technology and the effects of changes in the allocation of resources:*

$$\mathrm{d}\log \hat{Y}_c = \underbrace{\sum_{f \in F_c} \tilde{\Lambda}_f^{\hat{Y}_c} \,\mathrm{d}\log L_f + \sum_{i \in N - N_c} \tilde{\lambda}_i^{\hat{Y}_c} \,\mathrm{d}\log(-q_{ci}) + \sum_{i \in N_c} \tilde{\lambda}_i^{\hat{Y}_c} \,\mathrm{d}\log A_i}_{\Delta \mathit{Technology}}$$
$$\underbrace{- \sum_{i \in N_c} \tilde{\lambda}_i^{\hat{Y}_c} \,\mathrm{d}\log \mu_i - \sum_{f \in F_c}^{F} \tilde{\Lambda}_f^{\hat{Y}_c} \,\mathrm{d}\log \Lambda_f^{\hat{Y}_c}}_{\Delta \mathit{Reallocation}}.$$

The change $\mathrm{d}\log \hat{Y}$ *of world gross real output, which coincides with the change in world real output* $\mathrm{d}\log Y$*, can be obtained by simply suppressing the country index c.*

The main differences between Theorem 9 and its equivalent Theorem 1 for economies without distortions are as follows. First the "pure" technology effects use cost-based (and not revenue-based) exposures. Second, because we characterize changes in gross real output (and not real output), changes in imports show up as changes in factor supplies via the term $\sum_{i \in N - N_c} \tilde{\lambda}_i^{\hat{Y}_c} \,\mathrm{d}\log(-q_{ci})$. Third, there are non-zero reallocation effects. The term $-\sum_{f \in F_c}^{F} \tilde{\Lambda}_f^{Y_c} \,\mathrm{d}\log \Lambda_f^{Y_c}$ is a weighted average of the changes in the domestic factor income shares. When it is positive, it means that domestic factor shares are reduced on average, which, loosely speaking, means that the domestic share of profits is increasing. It indicates that resources are being reallocated to more distorted parts of the domestic economy, which increases real gross output because these parts of the domestic economy were too small to begin with from a social perspective. Of course, when markups/wedges increase, this mechanically increases the domestic profit share and reduces average domestic factor income shares. This effect must therefore be netted out and this is the role of the term $\sum_{i \in N_c} \tilde{\lambda}_i^{\hat{Y}_c} \,\mathrm{d}\log \mu_i$.

Following Baqaee and Farhi (2017b), we can define the aggregate productivity of country c via the "distorted" Solow residual

$$\mathrm{d}\log \hat{Y}_c = \sum_{f\in F_c} \tilde{\Lambda}_f^{\hat{Y}_c} \mathrm{d}\log L_f + \sum_{i\in N-N_c} \tilde{\lambda}_i^{\hat{Y}_c} \mathrm{d}\log(-q_{ci}).$$

The distorted Solow residual correctly accounts for the "pure" technology effect of changes in factor inputs by weighing them by their cost-based real gross output exposures (and not by their revenue-based gross real output exposures, as would be the case in the traditional Solow residual). In the presence of distortions, it is no longer true that the aggregate productivity of a country does not respond to shocks to productivities or iceberg trade cost (or to tariffs) outside of its borders, because of nonzero reallocation effects.

At the world level, we recover precisely the result of Baqaee and Farhi (2017b) for a closed economy:

$$\mathrm{d}\log Y = \mathrm{d}\log W = \sum_{i\in N} \tilde{\lambda}_i \mathrm{d}\log A_i - \sum_{i\in N} \tilde{\lambda}_i \mathrm{d}\log \mu_i - \sum_{f\in F} \tilde{\Lambda}_f \mathrm{d}\log \Lambda_f.$$

Theorem 10 (Welfare-Accounting, Reallocation)**.** *The change in welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be decomposed into the "pure" effects of changes in technology and the effects of changes in the allocation of resources:*

$$\mathrm{d}\log W_c = \underbrace{\sum_{f\in F} \tilde{\Lambda}_f^{W_c} \mathrm{d}\log L_f + \sum_{i\in N} \tilde{\lambda}_i^{W_c} \mathrm{d}\log A_i}_{\Delta Technology}$$
$$\underbrace{- \sum_{i\in N} \tilde{\lambda}_i^{W_c} \mathrm{d}\log \mu_i - \sum_{f\in F} \tilde{\Lambda}_f^{W_c} \mathrm{d}\log \Lambda_f + \sum_{f\in F^*} \Lambda_f^c \mathrm{d}\log \Lambda_f + (1/\chi_c^W)\, \mathrm{d}\, T_c}_{\Delta Reallocation}.$$

The change $\mathrm{d}\log W$ *in world real expenditure or welfare can be obtained by simply suppressing the country index c*

The main differences between Theorem 10 and its equivalent Theorem 3 for economies with no distortions are as follows. First, they use cost-based exposures rather than revenue-based exposures. Second, they account for the changes in the contributions to income of the revenues raised by the different markups/wedges, which is reflected in the sum over $f \in F^*$ (and not $f \in F$). Third, they account for the effects of changes in markups/wedges on the country welfare deflator, which is reflected in the term $-\sum_{i\in N} \tilde{\lambda}_i^{W_c} \mathrm{d}\log \mu_i$.[44]

[44] At the world level, where we recover once again the result of Baqaee and Farhi (2017b) for a closed

Theorems 9 and 10 give a unified framework for growth and productivity accounting in open, closed, distorted, and undistorted economies. They use changes in factor shares (ex-post sufficient statistics). We now supplement them with propagation equations which express changes in factor shares as a function of microeconomic primitives (ex-ante sufficient statistics).

Comparative Statics: Ex-Post Sufficient Statistics

We redefine the $(N+F) \times (N+F)$ "propagation-via-substitution" matrix Γ and the $(N+F) \times F^*$ "propagation-via-redistribution" matrix Ξ:

$$\Gamma_{ij} = \sum_{k \in N} (\theta_k - 1) \frac{\lambda_k / \mu_k}{\lambda_i} Cov_{\Omega^{(k)}} \left(\Psi_{(i)}, \tilde{\Psi}_{(j)} \right),$$

$$\Xi_{if} = \frac{1}{\lambda_i} \sum_{c \in C} (\lambda_i^{W_c} - \lambda_i) \Phi_{cf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor. The only differences with the case with no distortions are as follows. First, we now use a mix of revenue-based and cost-based columns of the Leontief inverse matrix $\Psi_{(i)}$ and $\tilde{\Psi}_{(j)}$, because the transmission of the transmission of expenditures is governed by Ψ and the transmission of prices by $\tilde{\Psi}$. Second, the sales share of producer k is divided by its markup/wedge μ_k because what matters is its cost, not its revenue. Third, redistribution terms are now also defined for fictitious factors.

Finally, we also define the $(N+F) \times N$ "propagation-via-input-demand-suppression" matrix Σ:

$$\Sigma_{ij} = (1_{\{i=j\}} - \frac{\lambda_j}{\lambda_i} \Psi_{ji}).$$

This matrix will play a role for the characterization of the changes in sales shares and factor shares in response to shocks to markups/wedges, because while these shocks act like negative productivity shocks on prices (for given factor wages) and their associated substitution effects, as captured by the propagation-via-substitution matrix, they also release resources via a reduction in the demand for inputs.

Theorem 11 (Factor Shares and Sales Shares)**.** *The changes in the sales share of goods and factors in response to a productivity shock to producer i are the solution of the following system of*

economy.

linear equations:[45]

$$\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log A_i}=\Gamma_{ji}-\sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i}+\sum_{g\in F^*}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i}\quad \textit{for } j\in N,$$

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i}=\Gamma_{fi}-\sum_{g\in F}\Gamma_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i}+\sum_{g\in F^*}\Xi_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_i}\quad \textit{for } f\in F,$$

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i}=\frac{\mathrm{d}\log\lambda_{\iota(f)}}{\mathrm{d}\log A_i}\quad \textit{for } f\in F^*-F.$$

The changes in the sales share of goods and factors in response to a markup/wedge shock to producer i are the solution of the following system of linear equations:[46]

$$\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log \mu_i}=\Sigma_{ji}-\Gamma_{ji}-\sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log \mu_i}+\sum_{g\in F^*}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log \mu_i}\quad \textit{for } j\in N,$$

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log \mu_i}=\Sigma_{fi}-\Gamma_{fi}-\sum_{g\in F}\Gamma_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log \mu_i}+\sum_{g\in F^*}\Xi_{fg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log \mu_i}\quad \textit{for } f\in F,$$

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log \mu_i}=\frac{\mathrm{d}\log\lambda_{\iota(f)}}{\mathrm{d}\log \mu_i}+1_{\{\iota(f)=i\}}\frac{1}{\mu_i-1}\quad \textit{for } f\in F^*-F.$$

More generally, we can use the characterization of the responses of sales shares and factor shares to characterize the responses of the input-output matrix, of the Leontief inverse matrix, and of all the income shares and all the exposures in real output and real expenditure or welfare, at the country and world levels, cost- and revenue-based.

Armed with Theorem 11, it is straightforward to characterize the response of prices and quantities to shocks.[47]

Corollary 11. *(Prices and Quantities) The changes in the wages of factors and in the prices and quantities of goods in response to a productivity shock to producer i are given by:*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_i}=\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_i},$$

[45]When $f\in F^*-F$, if $\mu_{\iota(f)}=1$, we have $\Lambda_f=0$, and so $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_i$ is not defined. The corresponding equation can then be omitted by using the convention $\Xi_{jf}\,\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_i=0$.

[46]When $f\in F^*-F$, if $\mu_{\iota(f)}=1$, we have $\Lambda_f=0$ and so $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log\mu_i$ is not defined. The corresponding equation can then be omitted by using the convention $\Xi_{jf}\,\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_i=0$. When $\mu_i=1$, $\mathrm{d}\log\mu_i$ is not defined, and so we cannot define the elasticities of sales shares with respect to μ_i, but we can define and compute their semi-elasticities in a straightforward way, but we omit the details for brevity. The same remark applies to Corollary 11.

[47]Recall that prices are expressed in the numeraire where $GDP=GNE=1$ at the world level.

$$\frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_i} = -\tilde{\Psi}_{ji} + \sum_{g\in F} \tilde{\Psi}_{jg} \frac{\mathrm{d}\log w_g}{\mathrm{d}\log A_i},$$
$$\frac{\mathrm{d}\log y_j}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log \lambda_j}{\mathrm{d}\log A_i} - \frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_k},$$

where $\mathrm{d}\log\Lambda_f / \mathrm{d}\log A_i$ *is given in Theorem 5. The changes in the wages of factors and in the prices and quantities of goods in response to a markup/wedge shock to producer i are given by:*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log \mu_i} = \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i},$$
$$\frac{\mathrm{d}\log p_j}{\mathrm{d}\log \mu_i} = \tilde{\Psi}_{ji} + \sum_{g\in F} \tilde{\Psi}_{jg} \frac{\mathrm{d}\log w_g}{\mathrm{d}\log \mu_i}$$
$$\frac{\mathrm{d}\log y_j}{\mathrm{d}\log \mu_i} = \frac{\mathrm{d}\log \lambda_j}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\log p_j}{\mathrm{d}\log \mu_i},$$

where $\mathrm{d}\log\Lambda_f / \mathrm{d}\log \mu_i$ *is given in Theorem 5.*

F Heterogenous Households Within Countries

To extend the mode to allow for a set of heterogenous agents $h \in H_c$ within countries $c \in C$, we proceed as follows. We denote by H the set of all households. Each household h in country c maximizes a homogenous-of-degree-one demand aggregator[48]

$$C_h = \mathcal{W}_h(\{c_{hi}\}_{i\in N}),$$

subject to the budget constraint

$$\sum_{i\in N} p_i c_{hi} = \sum_{f\in F} \Phi_{hf} w_f L_f + T_h,$$

where c_{hi} is the quantity of the good produced by producer i and consumed by the household, p_i is the price of good i, Φ_{hf} is the fraction of factor f owned by household, w_f is the wage of factor f, and T_h is an exogenous lump-sum transfer.

We define the following country aggregates: $c_{ci} = \sum_{h\in H_c} c_{hi}$, $\Phi_{cf} = \sum_{h\in H_c} \Phi_{hf}$, and $T_c = \sum_{h\in H_c} T_h$. We also define the HAIO matrix at the household level as a $(H + N +$

[48]In mapping our model to data, we interpret domestic "households" as any agent which consumes resources without producing resources to be used by other agents. Specifically, this means that we include domestic investment and government expenditures in our definition of "households" when we map this model to the data.

$F) \times (H + N + F)$ matrix Ω and the Leontief inverse matrix as $\Psi = (I - \Omega)^{-1}$.

All the definitions in Section 2 go through. In addition, we introduce the corresponding household-level definitions for a household h. First, the nominal output and the nominal expenditure of the household are:

$$GDP_h = \sum_{f \in F} \Phi_{hf} w_f L_f, \quad GNE_h = \sum_{i \in N} p_i c_{hi} = \sum_{f \in F} \Phi_{hf} w_f L_f + T_h,$$

where we think of the household as a set producers intermediating the uses by the different producers of the different factor endowments of the household. Second, the changes in real output and real expenditure or welfare of the household are:

$$\mathrm{d}\log Y_h = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log L_f, \quad \mathrm{d}\log P_{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log w_f,$$

$$\mathrm{d}\log W_h = \sum_{i \in N} \chi_i^{W_h} \,\mathrm{d}\log c_{hi}, \quad \mathrm{d}\log P_{W_h} = \sum_{i \in N} \chi_i^{W_h} \,\mathrm{d}\log p_i,$$

with $\chi_f^{Y_h} = \Phi_{hf} w_f l_f GDP_h$ and $\chi_i^{W_h} = p_i c_{hi} / GNE_h$. Third, the exposure to a good or factor k of the real expenditure and real output of household h is given by

$$\lambda_k^{W_h} = \sum_{i \in N} \chi_i^{W_h} \Psi_{ik}, \quad \lambda_k^{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \Psi_{fk},$$

where recall that $\chi_i^{W_h} = p_i c_{hi} / GNE_h$ and $\chi_f^{Y_h} = \Phi_{hf} w_f L_f / GDP_h$. The exposure in real output to good or factor k has a direct connection to the sales of the producer:

$$\lambda_k^{Y_h} = 1_{\{k \in F\}} \frac{\Phi_{hk} p_k y_k}{GDP_h},$$

where $\lambda_k^{Y_h} = 1_{\{k \in F\}} \Phi_{hk} (GDP/GDP_h) \lambda_k$ the local Domar weight of k in household h and where $\Phi_{hk} = 0$ for $k \in N$ to capture the fact that the household endowment of the goods are zero. Fourth, the share of factor f in the income or expenditure of the household is given by

$$\Lambda_f^h = \frac{\Phi_{hf} w_f L_f}{GNE_h}.$$

The results in Section 3 go through without modification. Theorems 1, 2, and 3, as well as Corollary 1 can be extended to the level of a household h by simply replacing the country index c by the household index h.

The results in Section 5 go through with the following modifications. The $(N + F) \times$

$(N+F)$ propagation-via-substitution matrix Γ must now be defined as

$$\Gamma_{ij} = \sum_{k \in N} (\theta_k - 1) \frac{\lambda_k}{\lambda_i} Cov_{\Omega^{(k)}} \left(\Psi_{(i)}, \Psi_{(j)} \right),$$

and the $(N+F) \times F$ propagation-via-redistribution matrix Ξ as

$$\Xi_{if} = \sum_{h \in H} \frac{\lambda_i^{W_h} - \lambda_i}{\lambda_i} \Phi_{hf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor.

The results in Section 6.3 go through with the following changes. Theorem 6 go through without modification, and be extended at the household level where $\Delta \log Y_h \approx 0$.

Corollary 6 goes through with some minor modifications. The world Bergson-Samuelson welfare function is now $W^{BS} = \sum_h \overline{\chi}_h^W \log W_h$,, changes in world welfare are measured as $\Delta \log \delta$, where δ solves the equation $W^{BS}(\overline{W}_1, \ldots, \overline{W}_H) = W^{BS}(W_1/\delta, \ldots, W_H/\delta)$, where $\overline{W}_h$ and W_h are the values at the initial efficient equilibrium. We use a similar definition for country level welfare δ_c, and the same notation for household welfare δ_h. Changes in world welfare are given up to the second order by

$$\Delta \log \delta \approx \Delta \log W + Cov_{\chi_h^W} \left(\Delta \log \chi_h^W, \Delta \log P_{W_h} \right),$$

, changes in country welfare are given up to the first order by

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c},$$

and the change in country welfare up to the first order by

$$\Delta \log \delta_h \approx \Delta \log W_h \approx \Delta \log \chi_h^W - \Delta \log P_{W_h}.$$

Theorems 6 and Corollary 6 go through with some minor modifications. Changes in factor shares are given up to the first order by the system of linear equations

$$\Delta \log \Lambda_f \approx - \sum_{i \in N} \Gamma_{fi} \Delta \log \mu_i - \sum_{g \in F} \Gamma_{fg} \Delta \log \Lambda_g + \sum_{g \in F} \Xi_{fg} \Delta \log \Lambda_g - \sum_{i \in N} \frac{\lambda_i}{\Lambda_f} \Psi_{if} \Delta \log \mu_i + \sum_{i \in N} \Xi_{fi} \Delta \log \mu_i,$$

where the definition of Ξ is extended for $f \in F$ and $i \in N$ by $\Xi_{fi} = \frac{1}{\Lambda_f} \sum_{h \in H} (\Lambda_f^{W_h} - \Lambda_f) \Phi_{hi} \lambda_i$, and Φ_{hi} is the share of the revenue raised by the tariff or other distortion on good i which accrues to household h. Changes in household income shares are given up to the first order by

$$\chi_h^W \Delta \log \chi_h^W = \sum_{g \in F} \Phi_{hf} \Lambda_g \Delta \log \Lambda_g + \sum_{i \in N} \Phi_{hi} \lambda_i \Delta \log \mu_i,$$

and changes in country income shares are given by $\Delta \log \chi_c^W = \sum_{h \in H_c} \chi_h^{W_c} \Delta \log \chi_h^W$. Changes in household real expenditure deflators are given up to the first order by

$$\Delta \log P_{W_h} = \sum_{i \in N} \lambda_i^{W_h} \Delta \log \mu_i + \sum_{g \in F} \Lambda_g^{W_h} \Delta \log \Lambda_g.$$

In Theorem 6, changes in world real output and real expenditure are given up to the second order by

$$\begin{aligned}
\Delta \log Y = \Delta \log W \approx -\frac{1}{2} \sum_{l \in N} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}} (\Psi_{(k)}, \Psi_{(l)}) \\
- \frac{1}{2} \sum_{l \in N} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}} (\Psi_{(g)}, \Psi_{(l)}) \\
+ \frac{1}{2} \sum_{l \in N} \sum_{h \in H} \chi_h^W \Delta \log \chi_h^W \Delta \log \mu_l (\lambda_l^{W_h} - \lambda_l),
\end{aligned}$$

changes in the real output of country c are given up to the second order by

$$\begin{aligned}
\Delta \log Y_c \approx -\frac{1}{2} \sum_{l \in N_c} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j^{Y_c} \theta_j Cov_{\Omega^{(j)}} (\Psi_{(k)}, \Psi_{(l)}) \\
- \frac{1}{2} \sum_{l \in N_c} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j^{Y_c} \theta_j Cov_{\Omega^{(j)}} (\Psi_{(g)}, \Psi_{(l)}) \\
+ \frac{1}{2} \sum_{l \in N_c} \sum_{h \in H} \chi_h^W \Delta \log \chi_h^W \Delta \log \mu_l (\lambda_l^{W_h} - \lambda_l) / \chi_c^Y,
\end{aligned}$$

and changes in the real output are given up to the second order by $\Delta \log Y_h \approx 0$. Corollary 7 goes through unchanged and can also be applied at the level of a household, using Corollary 6.

G Ex-Ante Comparative Statics for Transfer Shocks

Define the $(N+F)\times C$ matrix Ξ^T:

$$\Xi^T_{ic} = \sum_{c\in C} \frac{\lambda_i^{W_c}}{\lambda_i}.$$

For some feasible perturbation of transfers $\sum_{i\in C} \mathrm{d}\, T_c = 0$, changes in factor shares solve the following system of linear equations

$$\mathrm{d}\log\Lambda_f = -\sum_{g\in F}\Gamma_{fg}\,\mathrm{d}\log\Lambda_g + \sum_{g\in F}\Xi_{fg}\,\mathrm{d}\log\Lambda_g + \Xi^T_{fc}\,\mathrm{d}\, T_c.$$

Changes in sales shares are then given by

$$\mathrm{d}\log\lambda_j = -\sum_{g\in F}\Gamma_{jg}\,\mathrm{d}\log\Lambda_g + \sum_{g\in F}\Xi_{jg}\,\mathrm{d}\log\Lambda_g + \Xi^T_{jc}\,\mathrm{d}\, T_c.$$

H Roy Models

Galle et al. (2017)(GRY) combine a Roy-model of labor supply with an Eaton-Kortum model of trade to study the effects of trade on different groups of workers in an economy. They prove an extension to the ACR result that accounts for the distributional consequences of trade shocks. In this section, we show how our framework can be adapted for analyzing models such models. We generalize our analysis to encompass Roy-models of the labor market, and show how duality with the closed economy can then be used to study the distributional consequences of trade.

Suppose that H_c denotes the set of households in country c. As in Galle et al. (2017), households consume the same basket of goods, but supply labor in different ways.[49] We assume that each household type has a fixed endowment of labor L_h, which are assigned to work in different industries according to the productivity of workers in that group and the relative wage differences offered in different industries.

Define δ_{hf} to be type h's share of income derived from earning wages f

$$\delta_{hf} = \frac{\Phi_{hf}\Lambda_f}{\chi_h}.$$

[49]Similar methods can also be used when households have different consumption baskets, although Borusyak and Jaravel (2018) suggest that at least in the US, households have similar imported consumption baskets.

Letting world GDP be the numeraire, then the Roy model implies that

$$\Lambda_f = \sum_{h \in H} \delta_{hf} \chi_h \left(\frac{w_f}{\chi_h / L^h} \right)^{\varepsilon_h} \chi_h,$$

where ε_h is the supply elasticity and L^h is a shock to the stock of labor h has been endowed with (since we analyze log changes, the value of the endowment is irrelevant). The Roy model implies that

$$\chi_h = \left(\sum_f \delta_{hf} w_f^{\varepsilon_h} \right)^{\frac{1}{\varepsilon_h}} L^h,$$

where $\varepsilon_h = 1$ represents the case where labor cannot be moved across markets by h. If $\varepsilon_h > 1$ then h can take advantage of wage differentials to redirect its labor supply and boost its income. When $\varepsilon \to \infty$, labor mobility implies that all wages in the economy are equalized (and the model collapses to a one-factor model). Galle et al. (2017) show that the above equations can be microfounded via a ROY model where homogenous workers in each group type draw their ability for each job from Frechet distributions, and choose to work in the job that offers them the highest return.

Proposition 12. *The change in the welfare of group $g \in H_c$, in response to iceberg trade shocks is given by*

$$\Delta \log W_g = -\log \left(\frac{\check{P}_g(\check{A}, \check{w})}{\check{P}_c(\overline{\check{A}}, \check{w})} \right).$$

where

$$\Delta \log \check{A}_i = \log \left(\frac{\check{A}_i}{\overline{\check{A}}_i} \right) = \frac{1}{\zeta_i - 1} \log \left(\frac{\sum_{j \in N_c} \Omega_{ij}}{\sum_{j \in N_c} \overline{\Omega}_{ij}} \right),$$

and

$$\Delta \log \check{w}_f = \log \left(\frac{\check{w}_f}{\overline{\check{w}}_f} \right) = \frac{1}{\varepsilon_g} \log \left(\frac{\delta_{gf}}{\overline{\delta}_{gf}} \right).$$

In the case where $\varepsilon \to \infty$, we recover the one-factor model.

Of course, due to the fact that factor shares δ_{gf} are now endogenously responding to factor prices, the results in Baqaee and Farhi (2017a) can no longer be used to determine how these shares will change in equilibrium. Therefore, we extend those propositions here.

Proposition 13. *The response of the factor prices to a shock* $\mathrm{d} \log A_k$ *is the solution to the following system:*

1. *Product Market Equilibrium:*

$$\Lambda_l \frac{\mathrm{d}\log \Lambda_l}{\mathrm{d}\log A_k} = \sum_{i \in \{H,N\}} \lambda_j (1-\theta_j) Cov_{\Omega^{(j)}} \left(\Psi_{(k)} + \sum_f \Psi_{(f)} \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)} \right)$$
$$+ \sum_{h \in H} (\lambda_l^h - \lambda_l) \left(\sum_{f \in F_c} \Phi_{hf} \Lambda_f \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} \right).$$

2. *Factor Market Equilibrium:*

$$\mathrm{d}\log \Lambda_f = \sum_{h \in H} E_{\Phi^{(h)}} \left[\varepsilon_h \left(E_{\delta^{(h)}} \left(\mathrm{d}\log w_f - \mathrm{d}\log w \right) \right) + \left(E_{\delta^{(h)}} (\mathrm{d}\log w) \right) + (\mathrm{d}\log L) \right].$$

Given this, the welfare of the gth group is

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s \in F} \left(\frac{\Phi_{gs}}{\chi_g} \Lambda_s - \Lambda_s^g \right) \mathrm{d}\log w_s + \lambda_k^g + \mathrm{d}\log L^g.$$

I Welfare Decompositions: Simple Illustrative Examples

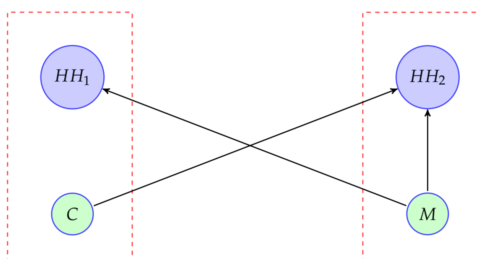

Figure 5: An illustration of the two welfare decompositions in an economy with two countries, two factors, and two goods. Country 1 has an endowment of a commodity good (C), and country 2 has an endowment of the manufacturing good (M). The representative household in country 1 consumes only the manufacturing good, and the representative household in country 2 consumes a CES aggregate of the two goods with an elasticity of substitution θ.

Fourth, the two decompositions have different economic interpretations. It is useful to provide a simple illustrative example. Consider the economy depicted in Figure 5 with two countries, two factors, and two goods. Country 1 has an endowment of a commodity good (C), and country 2 has an endowment of the manufacturing good (M). The representative household in country 1 consumes only the manufacturing good, and the representative household in country 2 consumes a CES aggregate of the two goods with an elasticity of substitution θ:

$$\left(\bar{\omega}_{2C}(\frac{y_{2C}}{\bar{y}_{2C}})^{\frac{\theta-1}{\theta}} + \bar{\omega}_{2M}(\frac{y_{2M}}{\bar{y}_{2M}})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

C and M can either be substitutes ($\theta > 1$) or complements ($\theta < 1$). We denote by λ_2 the sales share of the consumption bundle of producer 2, and by Λ_C and Λ_M the sales shares of C and M (the factor income shares), with $\lambda_2 \bar{\omega}_{2C} = \Lambda_C$.

Consider a shock $\mathrm{d}\,\bar{\omega}_{2M} = -\,\mathrm{d}\,\bar{\omega}_{2C} > 0$ which shifts the composition of demand away from C and towards M in country 2. [50] The shock reduces the welfare of country 1 with

$$\mathrm{d}\log W_1 = -\theta \frac{1}{\Lambda_M} \mathrm{d}\log \bar{\omega}_{2C} < 0.$$

There are neither real output nor "pure" technology effects, and there are equivalent negative terms-of-trade effects and reallocation effects:

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = \mathrm{d}\log \Lambda_C - \mathrm{d}\log \Lambda_M = -\theta \frac{1}{\Lambda_M} \mathrm{d}\log \bar{\omega}_{2C} < 0.$$

This can be seen as a simple illustration of the Prebisch-Singer hypothesis, whereby demand shifts towards manufacturing as countries develop at the expense of commodity producers.

Consider next a shock $\mathrm{d}\log C > 0$ which increases the endowment of C in country 1. The effect of the shock is different depending on whether C and M are substitutes (complements): it improves (reduces) the welfare of country 1 with

$$\mathrm{d}\log W_1 = (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M} \mathrm{d}\log C;$$

there are positive real output effects $\mathrm{d}\log Y_1 = \Lambda_C \,\mathrm{d}\log C > 0$ and less (more) negative terms-of-trade effects

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = -\Lambda_C \,\mathrm{d}\log C + (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M} \mathrm{d}\log C;$$

there are no "pure" technology effects, and positive (negative) reallocation effects

$$\mathrm{d}\log \Lambda_C - \mathrm{d}\log \Lambda_M = (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M} \mathrm{d}\log C.$$

[50] This shock can be modeled as a combination of positive and negative productivity shocks $\mathrm{d}\log A_{2M} = [\theta/(\theta-1)]\,\mathrm{d}\log \mathrm{d}\,\bar{\omega}_{2M}$ and $\mathrm{d}\log A_{2C} = [\theta/(\theta-1)]\,\mathrm{d}\log \mathrm{d}\,\bar{\omega}_{2C}$ for fictitious producers intermediating between C, M, and the representative household of country 2.

Finally, consider a shock which increases the endowment of M in country 2. This shock improves the welfare of country 1 as long as goods are not too substitutes with

$$\mathrm{d}\log W_1 = \mathrm{d}\log M - (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M;$$

; there are no real output effects and positive terms-of-trade effects as long as goods are not too substitutes with

$$\mathrm{d}\log p_C - \mathrm{d}\log p_M = \mathrm{d}\log M - (\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M;$$

there are positive "pure" technology effects $\mathrm{d}\log M > 0$ and negative (positive) reallocation effects if C and M are substitutes (complements) with

$$\mathrm{d}\log \Lambda_C - \mathrm{d}\log \Lambda_M = -(\theta - 1)\frac{\bar{\omega}_{2M}}{\Lambda_M}\,\mathrm{d}\log M.$$

J More on the Trade Elasticity

J.1 Necessary and Sufficient Conditions for Constant Trade Elasticity

We say that a good k is *relevant* for $\varepsilon(i, j; k)$ if

$$\lambda_m Cov_{\Omega^{(m)}}(\Psi_{(k)}, \Psi_{(i)}/\lambda_i - \Psi_{(j)}/\lambda_j) \neq 0.$$

If k is not relevant, we say that it is irrelevant. For instance, if some producer m is exposed symmetrically to i and j through its inputs

$$\Omega_{ml}(\Psi_{li} - \Psi_{lj}) = 0 \quad (l \in N),$$

then $\varepsilon(i, j; k)$ is not a function of θ_m and m is irrelevant. Another example is if some producer $m \neq j$ is not exposed to k through its inputs

$$\Psi_{mk} = 0,$$

then $\varepsilon(i, j; k)$ is not a function of θ_m and m is irrelevant.

Corollary 12 (Constant Trade Elasticity)**.** *Consider two distinct goods i and j that are imported to some country c. Then consider the following conditions:*

(i) Both i and j are unconnected to one another in the production network: $\Psi_{ij} = \Psi_{ji} = 0$*, and i is not exposed to itself* $\Psi_{ii} = 1$.

(ii) The representative "world" household is irrelevant

$$Cov_{\chi}\left(\Psi_{(i)}, \frac{\Psi_{(i)}}{\lambda_i} - \frac{\Psi_{(j)}}{\lambda_j}\right) = 0,$$

which holds if both i and j are only used domestically, so that only household c is exposed to i and j. That is, $\lambda_i^h = \lambda_j^h = 0$ *for all* $h \neq c$.

(iii) For every relevant producer l, the elasticity of substitution $\theta_l = \theta$.

The trade elasticity of i relative to j with respect to iceberg shocks to i is constant, and equal to

$$\varepsilon(i, j; i) = (\theta - 1).$$

if, and only if, (i)-(iii) hold.

The conditions set out in the example above, while seemingly stringent, actually represent a generalization of the conditions that hold in gravity models with constant trade elasticities. Those models oftentimes either assume away the production network, or assume that traded goods always enter via the same CES aggregator.

A noteworthy special case is when *i* and *j* are made directly from factors, without any intermediate inputs. Then, we have the following

Corollary 13. *(Network Irrelevance) If* λ_{ij} *and* λ_{ii} *are only made from domestic factors, then*

$$\sum_{m \in C,N} \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(ij)}, \Psi_{(ij)}/\lambda_{ij} - \Psi_{(ii)}/\lambda_{ii}) = 1.$$

Hence, if all microeconomic elasticities of substitution θ_m *are equal to the same value* $\theta_m = \theta$ *then* $\varepsilon_{ij,ij} = \theta$.

In the special case where *ij* and *ii* are both made only from domestic factors (no intermediate inputs are permitted), then the trade elasticity of imports from *j* to *i* with respect to trade costs between *i* and *j* become convex combinations of the underlying microelasticities. Of course, whenever all micro-elasticities of substitution are the same, the weights then become irrelevant, and this is the situation in most benchmark trade models with constant trade elasticities.

J.2 Trade Reswitching

Yi (2003) shows that the trade elasticity can be nonlinear due to vertical specialization, where the trade elasticity can increase as trade barriers are lowered. To see this, imagine there are two ways of producing a given good: the first technique uses a domestic supply chain and the other technique uses a global value chain. Whenever the good is domestically produced, the iceberg costs of transporting the good are, at most, incurred once — when the finished good is shipped to the destination. However, when the good is made via a global value chain, the iceberg costs are incurred as many times as the good is shipped across borders. As a function of the iceberg cost parameter τ, the difference in the price of these two goods (holding factor prices fixed) is a polynomial of the form

$$B_n \tau^n - B_1 \tau, \tag{3}$$

where B_n and B_1 are some coefficients and n is the number of times the border is crossed. The nonlinearity in τ, whereby the iceberg cost's effects are compounded by crossing the border, drives the sensitivity of trade volume to trade barriers in Yi (2003). The benefits from using a global value chain are compounded if the good has to cross the border many times.

However, this discussion indicates the behavior of the trade elasticity can, in principle, be much more complicated. In fact, an interesting connection can be made between the behavior of the trade elasticity and the (closed-economy) reswitching debates of the 1950s and 60s. Specifically, equation (3) is just one special case. In general, the cost difference between producing goods using supply chains of different lengths is a polynomial in τ – and this polynomial can, in principle, have more than one root. This means that the trade elasticity can be non-monotonic as a function of the trade costs, in fact, it can even have the "wrong" sign, where the volume of trade decreases as the iceberg costs fall. This mirrors the apparent paradoxes in capital theory where the relationship between the capital stock and the return on capital can be non-monotonic, and an increase in the interest rate can cause the capital stock to increase.[51]

[51] To see this in the trade context, imagine two perfectly substitutable goods, one of which is produced by using 10 units of foreign labor, the other is produced by shipping 1 unit of foreign labor to the home country, back to the foreign country, and then back to the home country and combining it with 10 units of domestic labor. If we normalize both foreign and domestic wages to be unity, then the he costs of producing the first good is $10(1+\tau)$, whereas the cost of producing the second good is $(1+\tau)^3 + 10$, where τ is the iceberg trade cost. When $\tau = 0$, the first good dominates and goods are only shipped once across borders. When τ is sufficiently high, the cost of crossing the border is high enough that the first good again dominates. However, when τ has an intermediate value, then it can become worthwhile to produce the second good, which causes goods to be shipped across borders many times, thereby inflating the volume of trade.

Such examples are extreme, but they illustrate the point that in the presence of input-output networks, the trade elasticity even in partial equilibrium (holding factor prices constant) can behave quite unlike any microeconomic demand elasticity, sloping upwards when, at the microeconomic level, every demand curve slopes downwards.

Non-Symmetry and Non-Triviality of Trade Elasticities

Another interesting subtlety of the Proposition **??** is that the aggregate trade elasticities are non-symmetric. That is, in general $\varepsilon_{ij,kl} \neq \varepsilon_{ji,kl}$. Furthermore, unlike the standard gravity equation, Proposition **??** shows that the cross-trade elasticities are, in general, nonzero. Hence, changes in trade costs between k and l can affect the volume of trade between i and j *holding fixed* relative factor prices and incomes. This is due to the presence of global value chains, which transmit shocks in one part of the economy to another independently of the usual general equilibrium effects (which work through the price of factors).

K Proofs

Proof of Theorem 1. For some country h, let N_h be the set of domestically produced goods and let N_{-h} be the set of foreign-produced goods. Let Ω^h be the matrix whose elements are

$$\Omega^h_{ij} = \frac{M_{ij}}{R_i} \quad (i, j \in N_h),$$

where M_{ij} is the value of i's purchases from j and R_i is the revenues of i. Similarly, let Ω^{-h} be given by

$$\Omega^f_{ij} = \frac{M_{ij}}{R_i} \quad (i \in H, j \in N_{-h}).$$

Finally, let

$$\alpha_{ij} = \frac{w_j L_{ij}}{R_i} \quad (i \in H),$$

where $w_j L_{ij}$ are factor payments by i to factor j.

Denoting the vector of domestic prices by p^h, foreign prices by p^{-h}, domestic productivity shocks and wages by w^h and A^h, Shephard's lemma implies that

$$\mathrm{d}\log p^h = (I - \Omega^h)^{-1} \left(\Omega^{-h} \,\mathrm{d}\log p^{-h} + \alpha \,\mathrm{d}\log w^h - \mathrm{d}\log A^h \right).$$

Now, let γ be each good's share in GDP, so for domestically produced goods

$$\gamma_i^h = \frac{F_i + X_i}{GDP},$$

where F_i is final domestic use and X_i is exports. For imported goods

$$\gamma_i^{-h} = -\sum_{j \in N_h} \Omega_{ji} \lambda_j, \quad (i \in N_{-h})$$

By definition,

$$d\log Y_h = \Lambda'_{(h)}(d\log w^h + d\log L^h) - d\log P_{Y_h},$$

where L^h is the vector of quantities and $\Lambda_{(h)}$ the local Domar weights of domestic factors. This can be written as

$$d\log Y_h = \Lambda'_{(h)}(d\log w^h + d\log L^h) - (\gamma^h)' d\log p^h + (\gamma^{-h})' d\log p^{-h}.$$

Substituting the expression for $d\log p^h$ gives

$$\begin{aligned} d\log Y_h = \Lambda'_{(h)}(d\log w + d\log L) - (\gamma^h)'(I - \Omega^h)^{-1} \left(\Omega^f d\log p^f + \alpha\, d\log w - d\log A\right) \\ + (\gamma^f)' d\log p^f. \end{aligned}$$

To complete the proof, note that

$$(\gamma^h)'(I - \Omega^h)^{-1}\alpha = \Lambda_{(h)},$$

$$(\gamma^h)'(I - \Omega^h)^{-1}\Omega^{-h} = \gamma^{-h},$$

and

$$(\gamma^h)'(I - \Omega^h)^{-1} = \lambda_{(h)},$$

where $\lambda_{(h)}$ is the vector of local Domar weights for domestic producers. These expressions follow from market clearing. Combining these gives

$$d\log Y_h = \Lambda'_{(h)} d\log L^h + \lambda'_{(h)} d\log A.$$

■

Proof of Theorem 3. This follows as a corollary of Proposition **??** in Appendix E. ■

Proof of Proposition **??**. This follows as a corollary of Proposition **??** in Appendix E. ■

Proof of Corollary 5. By Shephard's lemma,

$$\mathrm{d}\log p_i = \sum_{j\in N} \Omega_{ij}\,\mathrm{d}\log p_j + \sum_{j\in F} \Omega_{ij}\,\mathrm{d}\log w_j - \mathrm{d}\log A_i.$$

Solve this system of equations in $\mathrm{d}\log p$ to get the desired result. ■

Lemma 14. *In general, for any f and g,*

$$\sum_m \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(f)}, \Psi_{(g)}) = -\lambda_f \lambda_g - Cov_\chi(\Psi_{(f)}, \Psi_{(g)}) + \Psi_{gf}\lambda_g + \Psi_{fg}\lambda_f - \lambda_f \mathbf{1}(f = g).$$

Consider a good k which does not use itself directly or indirectly. Then

$$\sum_{m\in C,N} \lambda_m Var_{\Omega^m}(\Psi_{(k)}) = \lambda_k(1-\lambda_k) - Var_\chi(\Psi_{(k)}).$$

Consider two goods which don't rely on each other, then

$$\sum_m \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(f)}, \Psi_{(g)}) = -\lambda_f \lambda_g - Cov_\chi(\Psi_{(f)}, \Psi_{(g)}).$$

Proof of Theorem 4. Here we assume that there is only one factor in the domestic economy and normalize its price to one.

Define the "fictitious domestic" IO matrix

$$\Omega^c_{ij} \equiv \frac{\Omega_{ij}}{\sum_{k\in C} \Omega_{ik}},$$

with associated Leontief-inverse matrix

$$\Psi^c \equiv (1 - \Omega^c)^{-1}.$$

Then for each producer $i \in C$, we have

$$d\log p_i = \sum_{j\in C} \Omega^c_{ij} d\log p_j + \frac{d\log \lambda_{ic}}{\theta_i - 1},$$

where λ_{ic} is the domestic cost share of producer i. The solution of this system of equations is

$$d\log p_i = \sum_{j\in C} \Psi^c_{ij} \frac{d\log \lambda_{jc}}{\theta_j - 1}.$$

From this we can get welfare gains

$$d\log Y^c = -\sum_{i\in C} b_i^c d\log p_i = -\sum_{i\in C}\sum_{j\in C} b_i^c \Psi_{ij}^c \frac{d\log\lambda_{jc}}{\theta_j - 1} = -\sum_{j\in C}\lambda_j^c \frac{d\log\lambda_{jc}}{\theta_j - 1},$$

where

$$\lambda_i^c \equiv \sum_{j\in C} b_j^c \Psi_{ji}^c.$$

This can be thought of as hitting the fictitious domestic economy with productivity shocks $-d\log\lambda_{jc}/(\theta_j - 1)$. ■

Proof of Theorem 6. Proof of Part(1):

The expression for $\mathrm{d}^2\log Y$ follows from applying part (2) to the whole world. The equality of real GDI and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then, we can write:

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log\mu_i} = \sum_{f\in F_c}\Lambda_{cf}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} + \sum_j \frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_i}\frac{\left(1-\frac{1}{\mu_j}\right)}{P_{Y_c}Y_c} + \frac{\lambda_{ci}}{\mu_i} - \frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log\mu_i},$$

where

$$\frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log\mu_i} = \sum_{f\in F_c}\tilde{\Lambda}_{cf}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} + \sum_{m\in M_c}\tilde{\lambda}_{cm}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_i} - \tilde{\lambda}_{ci} - \sum_{m\in M_c}\Lambda_{cm}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_i},$$

and

$$\tilde{\lambda}_{ci} = \sum_j F_{cj}\tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned}\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log\mu_i} &= \sum_{f\in F_c}\left(\Lambda_{cf}-\tilde{\Lambda}_{cf}\right)\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} + \sum_{m\in M_c}\left(\lambda_{cm}-\tilde{\lambda}_{cm}\right)\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_i}\\ &\quad + \sum_{j\in N_c}\lambda_{cj}\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log\mu_i}\left(1-\frac{1}{\mu_j}\right) + \frac{\lambda_{ci}}{\mu_i} - \tilde{\lambda}_{ci}.\end{aligned}$$

At the efficient point,

$$\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} = \sum_{f\in F_c}\left(\frac{\mathrm{d}\,\Lambda_{cf}}{\mathrm{d}\log\mu_i} - \frac{\mathrm{d}\,\tilde{\Lambda}_{cf}}{\mathrm{d}\log\mu_i}\right)\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}$$

$$
+\sum_{m\in M_c}\left(\frac{\mathrm{d}\,\lambda_{cm}}{\mathrm{d}\log\mu_i}-\frac{\mathrm{d}\,\tilde{\lambda}_{cm}}{\mathrm{d}\log\mu_i}\right)\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_{ck}}{\mathrm{d}\log\mu_i}
$$

$$
+\lambda_{ck}\left(\frac{\mathrm{d}\log\lambda_{ck}}{\mathrm{d}\log\mu_i}-\delta_{ki}\right)+\frac{1}{P_{Y_c}Y_c}\frac{\mathrm{d}\,\lambda_{ci}}{\mathrm{d}\log\mu_k}
$$

Using Lemma 16,

$$
\begin{aligned}
\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}&=-\sum_{f\in F_c}\lambda_{ci}\Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}-\sum_{m\in M_c}\lambda_{ci}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\lambda_{ci}\left(\Psi_{ik}-\delta_{ik}\right)\\
&-\lambda_{ck}\delta_{ik}+\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{P_{Y_c}Y_c},\\
&=-\sum_{f\in F_c}\lambda_{ci}\Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}-\sum_{m\in M_c}\lambda_{ci}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\lambda_{ci}\Psi_{ik}\\
&+\lambda_{ci}\left(\frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k}+\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}\right),\\
&=\lambda_{ci}\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}.
\end{aligned}
$$

Proof of Part(3):

$$
\frac{\mathrm{d}\log W^{BS}}{\mathrm{d}\log\mu_k}=\sum_h\overline{\chi}_h\frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k}=\sum_h\overline{\chi}_h\left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right).
$$

$$
\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_k}=\sum_{f\in F_c}\frac{\Lambda}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}+\sum_{i\in N_h}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{(1-\frac{1}{\mu_i})}{\chi_h}.
$$

$$
\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}=\sum_{f\in F}\tilde{\Lambda}_f^h\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}+\tilde{\lambda}_k^h.
$$

Hence, assuming the normalization $P_Y Y=1$ gives

$$
\begin{aligned}
\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i}=\sum_h\overline{\chi}_h\Bigg(&\sum_f\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{1}{\chi_h}+\sum_f\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\\
&-\sum_f\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}+\frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i}\frac{1}{\chi_h\mu_k}-\frac{\lambda_k}{\chi_h\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}-\frac{\lambda_k}{\chi_h\mu_k}\delta_{ki}\\
&\sum_i\frac{\mathrm{d}^2\,\lambda_j}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h}+\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{\mu_i\chi_h}+\sum_j\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\
&-\sum_f\frac{\mathrm{d}\,\tilde{\Lambda}_f^h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}-\sum_f\tilde{\Lambda}_f^h\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_k^h}{\mathrm{d}\log\mu_i}\Bigg).
\end{aligned}
$$

At the efficient point, this simplifies to

$$
\begin{aligned}
\frac{\mathrm{d}^2 \log W^{BS}}{\mathrm{d}\log \mu_k \,\mathrm{d}\log \mu_i} = &\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log \mu_i} - \sum_h \overline{\chi_h}\frac{\mathrm{d}\,\tilde{\Lambda}_f^h}{\mathrm{d}\log \mu_i}\right) \\
&+ \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log \mu_i} - \sum_h \overline{\chi_h}\frac{\mathrm{d}\,\tilde{\lambda}_k^h}{\mathrm{d}\log \mu_i} - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \lambda_k \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} - \lambda_k \delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k}.
\end{aligned}
$$

By Lemma 15, at the efficient point,

$$
\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_i} - \sum_h \overline{\chi}_h \frac{\mathrm{d}\,\tilde{\lambda}_j^h}{\mathrm{d}\log \mu_i} = \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_j^h - \lambda_i\left(\Psi_{ij} - \delta_{ij}\right).
$$

Whence, we can further simplify the previous expression to

$$
\begin{aligned}
\frac{\mathrm{d}^2 \log W^{BS}}{\mathrm{d}\log \mu_k \,\mathrm{d}\log \mu_i} = &\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^h - \lambda_i \Psi_{if}\right) \\
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^h - \lambda_i(\Psi_{ik} - \delta_{ik}) - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log \chi_h}\mathrm{d}\log \mu_i - \lambda_k \delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k}, \\
= &\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^h - \lambda_i \Psi_{if}\right) \\
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^h - \lambda_i \Psi_{ik} - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log \chi_h}\mathrm{d}\log \mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k},
\end{aligned}
$$

and using $\mathrm{d}\log \lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$
\begin{aligned}
= &\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\Lambda}_f^h - \lambda_i \Psi_{if}\right) \\
&+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\tilde{\lambda}_k^h - \lambda_i \Psi_{ik} - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log \chi_h}\mathrm{d}\log \mu_i + \lambda_i \frac{\mathrm{d}\log p_i}{\mathrm{d}\log \mu_k} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k},
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{f,h} \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \tilde{\Lambda}_f^h \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \lambda_i \sum_f \Psi_{if} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \\
&+ \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_k^h - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \\
&- \lambda_k \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} \\
&+ \lambda_i \left(\sum_f \Psi_{if} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} + \Psi_{ik} \right), \\
&= \sum_{f,h} \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \left(\chi_h \tilde{\Lambda}_f^h - \Lambda_f \right) \\
&+ \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_k^h - \lambda_k \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \left(\tilde{\Lambda}_f^h \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} + \tilde{\lambda}_k^h \right) \\
&- \sum_{f,h} \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \Lambda_f - \lambda_k \sum_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log \mu_k} \\
&- \left(\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \Lambda_f \right) \left(\sum_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \right) - \lambda_k \sum_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k} + Cov_\chi \left(\frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log \mu_k} \right),
\end{aligned}
$$

since

$$
-\sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \Lambda_f = -\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log \mu_k} = \frac{\mathrm{d}\left(1 - \sum_j \lambda_j (1 - \frac{1}{\mu_j})\right)}{\mathrm{d}\log \mu_k} = -\lambda_k
$$

at the efficient point, and

$$
\sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} = 0.
$$

■

Lemma 15.

$$
\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_k} - \sum_h \overline{\chi}_h \frac{\mathrm{d}\log \tilde{\lambda}_j^h}{\mathrm{d}\log \mu_k} = \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_j^h - \lambda_i \left(\Psi_{ij} - \delta_{ij} \right).
$$

Proof. Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its

kth diagonal element. Then for each h,

$$\tilde{\lambda}^h = b^{(h)} + \tilde{\lambda}^h \mu \Omega.$$

Hence,

$$\frac{\mathrm{d}\,\tilde{\lambda}^h}{\mathrm{d}\log \mu_k} = \frac{b^{(h)}}{\mathrm{d}\log \mu_k} + \frac{\mathrm{d}\,\tilde{\lambda}^h}{\mathrm{d}\log \mu_k}\mu\Omega + \tilde{\lambda}^h I_{\mu_k}\Omega + \tilde{\lambda}^h \mu \frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k}.$$

Hence,

$$\overline{\chi}' \frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log \mu_k} = \chi' \frac{b}{\mathrm{d}\log \mu_k} + \chi' \frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log \mu_k}\mu\Omega + \chi'\tilde{\lambda} I_{\mu_k}\Omega + \chi'\tilde{\lambda}\mu \frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k}.$$

On the other hand,

$$\lambda = \chi' b + \lambda \Omega.$$

Form this, we have

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} = \frac{\mathrm{d}\,\chi'}{\mathrm{d}\log \mu_k} b + \chi' \frac{b}{\mathrm{d}\log \mu_k} + \lambda \frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k} + \frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k}\Omega.$$

Combining these two expressions

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) = \left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) \Omega + \frac{\mathrm{d}\,\chi}{\mathrm{d}\log \mu_k} b - \chi' \tilde{\lambda}^{(h)} I_{\mu_k} \Omega.$$

Rearrange this to get

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log \mu_k} b\Psi - \chi' \tilde{\lambda}^{(h)} I_{\mu_k} (\Psi - I),$$

or

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} - \overline{\chi}' \frac{\mathrm{d}\log \tilde{\lambda}}{\mathrm{d}\log \mu_k} \right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log \mu_k} b\Psi - \lambda I_{\mu_k} (\Psi - I).$$

■

Lemma 16. *At the efficient steady-state*

$$\frac{\mathrm{d}\,\lambda_{cj}}{\mathrm{d}\log \mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_{cj}}{\mathrm{d}\log \mu_k} = -\lambda_{ck} \left(\Psi_{kj} - \delta_{kj} \right).$$

Proof. Start from the relations

$$\lambda_{cj} = F_j^c + \sum_i \lambda_{ci} \Omega_{ij},$$

and

$$\tilde{\lambda}_{cj} = F_j^c + \sum_i \tilde{\lambda}_{ci}\mu_i\Omega_{ij}.$$

Differentiate both to get

$$\frac{\mathrm{d}\,\lambda_{cj}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_{cj}}{\mathrm{d}\log\mu_k} = \sum_i \left(\frac{\mathrm{d}\,\lambda_{cj}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_{cj}}{\mathrm{d}\log\mu_k}\right)\Omega_{ij} - \lambda_{ck}\Omega_{ki}.$$

Rearrange this to get the desired result. ■

Proof of Proposition **??**. To loglinearize real GDP of country c, let country c's nominal GDP be the numeraire. Then

$$\mathrm{d}\log Y_c = -\,\mathrm{d}\log P_{Y_c},$$

whence

$$\begin{aligned}
&= -\sum_{i\in N_c} F_{ci}\,\mathrm{d}\log p_i + \sum_{i\notin N_c}\lambda_{ci}\,\mathrm{d}\log p_i,\\
&= \sum_{i\in N_c} F_{ci}\sum_{j\in N_c}\tilde{\Psi}^{dd}_{ij}\left(\mathrm{d}\log A_j - \mathrm{d}\log\mu_j\right) - \sum_{f\in F_c}\sum_{i\in N_c} F_{ci}\sum_{j\in N_c}\tilde{\Psi}^{dd}_{ij}\tilde{\Omega}_{jf}\,\mathrm{d}\log w_f\\
&- \sum_{k\notin N_c}\sum_{i\in N_c} F_{ci}\sum_{j\in N_c}\tilde{\Psi}^{dd}_{ij}\tilde{\Omega}_{jk}\,\mathrm{d}\log p_k + + \sum_{i\notin N_c}\lambda_{ci}\,\mathrm{d}\log p_i.
\end{aligned}$$

where $\tilde{\Omega}^{dd}$ is the domestic-domestic submatrix of the (cost-based) input-output matrix.

This can be further simplified to

$$\begin{aligned}
\mathrm{d}\log Y_c &= \sum_{i\in N_c}\tilde{\lambda}_j\left(\mathrm{d}\log A_j - \mathrm{d}\log\mu_j\right) - \sum_{f\in F_c}\tilde{\Lambda}_{cf}\,\mathrm{d}\log w_f\\
&- \sum_{k\notin N_c}\tilde{\lambda}_{ck}\,\mathrm{d}\log p_k + \sum_{k\notin N_c}\lambda_{ck}\,\mathrm{d}\log p_k,\\
&= \sum_{i\in N_c}\tilde{\lambda}_{cj}\left(\mathrm{d}\log A_j - \mathrm{d}\log\mu_j\right) - \sum_{f\in F_c}\tilde{\Lambda}_{cf}\,\mathrm{d}\log w_f + \sum_{k\notin N_c}(\lambda_{ck} - \tilde{\lambda}_{ck})\,\mathrm{d}\log p_k,
\end{aligned}$$

with

$$\tilde{\lambda}_{ck} = \sum_{j\in N_c} F_{cj}\tilde{\Psi}^{dd}_{jk}, \quad (k\in N_c),$$

$$\tilde{\lambda}_{ci} = \frac{p_i c_{ci}}{P_{Y_c} Y_c} + \sum_{j\in N_c} F_{cj}\tilde{\Psi}^{dd}_{jk}\tilde{\Omega}_{ki}, \quad (i\notin N_c),$$

and

$$\tilde{\Lambda}_{cf} = \sum_{j \in N_c} F_{cj} \tilde{\Psi}^{dd}_{jk} \tilde{\Omega}_{kf}, \quad (f \in F_c).$$

To finish, note that under our choice of numeraire

$$\mathrm{d}\log w_f + \mathrm{d}\log L_f = \mathrm{d}\log \Lambda_{cf}, \quad (f \in F_c)$$

$$\mathrm{d}\log p_i + \mathrm{d}\log y_i = \mathrm{d}\log \lambda_{ci}, \quad (i \in N_c),$$

and

$$\mathrm{d}\log p_i + \mathrm{d}\log m_i = \mathrm{d}\log \lambda_{ci}, \quad (i \notin N_c),$$

where m_i is total quantity of imports of i. Substitute this into the previous expressions to get

$$\mathrm{d}\log Y_c - \sum_{f \in F_c} \tilde{\Lambda}_{cf}\, \mathrm{d}\log L_f = \sum_{i \in N_c} \tilde{\lambda}_{cj} \left(\mathrm{d}\log A_j - \mathrm{d}\log \mu_j\right) - \sum_{f \in F_c} \tilde{\Lambda}_{cf}\, \mathrm{d}\log \Lambda_f$$
$$+ \sum_{k \notin N_c} (\lambda_{ck} - \tilde{\lambda}_{ck}) \left(\mathrm{d}\log \lambda_{ck} - \mathrm{d}\log m_i\right).$$

■

Proof of Proposition **??**. For each good,

$$\lambda_i = \sum_c b_{ci} \chi_c + \sum_i \Omega_{ji} \lambda_j.$$

This means

$$\lambda_i\, \mathrm{d}\log \lambda_i = \sum_c \chi_c b_{ci}\, \mathrm{d}\log b_{ci} + \sum_j \Omega_{ji} \lambda_j\, \mathrm{d}\log \Omega_{ji} + \sum_j \Omega_{ji}\, \mathrm{d}\,\lambda_j + \sum_c b_{ci} \chi_c\, \mathrm{d}\log \chi_c.$$

Now, note that

$$\mathrm{d}\log b_{ci} = (1 - \theta_c) \left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right)$$

$$\mathrm{d}\log \Omega_{ji} = (1 - \theta_j) \left(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log \mu_j\right) - \mathrm{d}\log \mu_j$$

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c}\, \mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\mu_i}\, \mathrm{d}\log \mu_i.$$

$$\mathrm{d}\log p_i = \tilde{\Psi} \left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + \tilde{\Psi} \tilde{\alpha}\, \mathrm{d}\log \Lambda.$$

$$\mathrm{d}\log P_{y_c} = b' \tilde{\Psi} \left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + b' \tilde{\Psi} \tilde{\alpha}\, \mathrm{d}\log \Lambda.$$

For shock $\mathrm{d}\log\mu_k$, we have

$$\mathrm{d}\log b_{ci} = (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j b_{cj}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right).$$

$$\mathrm{d}\log\Omega_{ji} = (1-\theta_j)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) - \theta_j\,\mathrm{d}\log\mu_j.$$

Putting this altogether gives

$$\begin{aligned}
\mathrm{d}\lambda_l = &\sum_i\sum_c(1-\theta_c)\chi_c b_{ci}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j b_{cj}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\Psi_{il} \\
&+ \sum_i\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
&- \theta_k\lambda_k\sum_i \Omega_{ki}\Psi_{il} + \sum_c \chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\lambda_l = &\sum_c(1-\theta_c)\chi_c\sum_i b_{ci}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} - \left(\sum_i b_{ci}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\left(\sum_i b_{ci}\Psi_{il}\right) \\
&+ \sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\sum_i\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} - \left(\sum_i\tilde{\Omega}_{ji}\Psi_{il}\right)\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) \\
&- \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this further to get

$$\begin{aligned}
\mathrm{d}\lambda_l = &\sum_c(1-\theta_c)\chi_c Cov_{b^{(c)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right) \\
&+ \sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\sum_i\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
&- \left(\sum_i\tilde{\Omega}_{ji}\Psi_{il}\right)\left(\sum_i\tilde{\Omega}_{ji}\tilde{\Psi}_{ik} + \sum_i\tilde{\Omega}_{ji}\sum_f \Psi_{if}\,\mathrm{d}\log\Lambda_f\right) \\
&- \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c,
\end{aligned}$$

Using the input-output covariance notation, write

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c Cov_{b^{(c)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right) \\
& + \sum_j (1-\theta_j)\lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right) \\
& - (1-\theta_k)\lambda_k(\Psi_{kl} - \mathbf{1}(l=k)) - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c,
\end{aligned}$$

This then simplifies to give from the fact that $\sum_i b_{ci}\Psi_{il} = \lambda_l^c$:

$$\begin{aligned}
\lambda_l\,\mathrm{d}\log\lambda_l = & \sum_{j\in N,C} (1-\theta_j)\lambda_j\mu_j^{-1} Cov(\tilde{\Psi}_{(k)} + \sum_f^F \mathrm{d}\log\Lambda_f, \Psi_{(l)}) \\
& - \lambda_k\left(\Psi_{kl} - \mathbf{1}(k=l)\right) + \sum_c \chi_c\lambda_l^c\,\mathrm{d}\log\chi_c.
\end{aligned}$$

To complete the proof, note that

$$P_{y_c}Y_c = \sum_f w_f L_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)p_i y_i.$$

Hence,

$$\mathrm{d}(P_{y_c}Y_c) = \sum_{f\in c} w_f L_f\,\mathrm{d}\log w_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)p_i y_i\,\mathrm{d}\log(p_i y_i) + \sum_{i\in c}\frac{\mathrm{d}\left(1-\frac{1}{\mu_i}\right)}{\mathrm{d}\log\mu_i}p_i y_i\,\mathrm{d}\log\mu_i.$$

In other words, since $P_y Y = 1$, we have

$$\mathrm{d}\,\chi_c = \sum_{f\in c}\Lambda_f\,\mathrm{d}\log w_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)\lambda_i\,\mathrm{d}\log\lambda_i + \sum_{i\in c}\frac{\mathrm{d}\left(1-\frac{1}{\mu_i}\right)}{\mathrm{d}\log\mu_i}\lambda_i\,\mathrm{d}\log\mu_i.$$

Hence,

$$\mathrm{d}\log\chi_c = \sum_{f\in F_c^*}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f + \sum_{i\in c}\frac{\lambda_i}{\chi_c}\,\mathrm{d}\log\mu_i.$$

■

Proof of Proposition 13. Loglinearizing this, we get

$$\Lambda_f \, \mathrm{d}\log\Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \delta_{hf}\chi_h\varepsilon_h - \sum_{h\in H}(\varepsilon_h - 1)\chi_h\delta_{hf}\sum_{l\in F}\delta_{hl}\,\mathrm{d}\log w_l + \sum_{h\in H}\chi_h\delta_{hf}\,\mathrm{d}\log L^h.$$

We can put this back into familiar notation

$$\Lambda_f \, \mathrm{d}\log\Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Phi_{hf}\Lambda_f\varepsilon_h - \sum_{h\in H}(\varepsilon_h - 1)\Phi_{hf}\Lambda_f\sum_{l\in F}\frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l + \sum_{h\in H}\Phi_{hf}\Lambda_f\,\mathrm{d}\log L^h.$$

Simplify this to get

$$\mathrm{d}\log\Lambda_f = \mathrm{d}\log w_f \sum_{h\in H} \Phi_{hf}\varepsilon_h - \sum_{h\in H}(\varepsilon_h - 1)\Phi_{hf}\sum_{l\in F}\frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l + \sum_{h\in H}\Phi_{hf}\,\mathrm{d}\log L^h.$$

We can beautify this a bit as

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H}\varepsilon_h E_{\Phi^{(h)}}\left(E_{\delta^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \sum_{h\in H} E_{\Phi^{(h)}}\left(E_{\delta^{(h)}}(\mathrm{d}\log w)\right) + \sum_{h\in H} E_{\Phi^{(h)}}\left(\mathrm{d}\log L\right).$$

or

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\varepsilon_h\left(E_{\delta^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\delta^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

or

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[E_{\delta^{(h)}}\left(\varepsilon_h\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right) + (\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

The case with immobile labor is given by $\varepsilon_h = 1$ for every $h \in H$, in which case $\mathrm{d}\log w_f = \mathrm{d}\log\Lambda_f$. Combine this with demand for the factors to finish the characterization

$$\Lambda_l\frac{\mathrm{d}\log\Lambda_l}{\mathrm{d}\log A_k} = \sum_{i\in\{H,N\}}\lambda_j(1-\theta_j)Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)}\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right) + \sum_{h\in H}(\lambda_l^h - \lambda_l)\left(\sum_{f\in F_c}\Phi_{hf}\Lambda_f\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).$$

This means that we can also redo the welfare accounting and write

$$\mathrm{d}\log W_g = \mathrm{d}\log\chi_g - \mathrm{d}\log P_g^c,$$

where χ_g is the income of household g. This can be written as

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\delta_{gs} - \Lambda_s^g\right)\mathrm{d}\log w_s + \lambda_k^g\,\mathrm{d}\log A_k + \mathrm{d}\log L^g,$$

or

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\frac{\Phi_{gs}}{\chi_g}\Lambda_s - \Lambda_s^g\right)\mathrm{d}\log w_s + \lambda_k^g + \mathrm{d}\log L^g.$$

Using trick by Rodriguez-Clare and Feenstra, we note that

$$\mathrm{d}\log\chi_g = \mathrm{d}\log w_s + \frac{1}{\varepsilon_g}\,\mathrm{d}\log\delta_{gs}$$

for any s, and hence

$$\mathrm{d}\log\chi_g = \sum_f \overline{\Lambda}_f^g\left(\mathrm{d}\log w_f + \frac{1}{\varepsilon_g}\,\mathrm{d}\log\delta_{gf}\right),$$

where $\overline{\Lambda^g}$ and $\overline{\lambda}^g$ are the Domar weights under the closed-economy IO matrix. Then we can combine this with the fact that

$$\mathrm{d}\log P_g^c = \sum_f \overline{\Lambda}_f^g\,\mathrm{d}\log w_f + \sum_i \overline{\lambda}_i\frac{\mathrm{d}\log\lambda_{ii}}{\theta_i - 1}$$

to get

$$\mathrm{d}\log W_g = \sum_f \overline{\Lambda}_f^g\frac{\mathrm{d}\log\delta_{gs}}{\varepsilon_g} + \sum_i \overline{\lambda}_i^g\frac{\mathrm{d}\log\lambda_{ii}}{\theta_i - 1}.$$

■