Debt as Safe Asset*

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Abstract

The price of a safe asset reflects not only the expected discounted future cash flows but also future service flows, since retrading allows partial insurance of idiosyncratic risk in an incomplete markets setting. This lowers the issuers' interest burden and allows the government to run a permanent (primary) deficit without ever paying back its debt. As idiosyncratic risk rises during recessions, so does the value of the service flows bestowing the safe asset with a negative β. This resolves government debt valuation puzzles. Nevertheless, the government faces a "Debt Laffer Curve". The paper also has important implications for fiscal debt sustainability and the FTPL.

Keywords: Safe Asset, Government Debt, Debt Laffer Curve, Ponzi Scheme, FTPL, Fiscal Capacity, I Theory of Money, r vs. g

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1 Introduction

How much government debt can the market absorb? At what interest rate? Is there a limit, a "Debt Laffer Curve"? When can governments run a permanent (primary) deficit without ever paying back its debt, like a Ponzi scheme, and nevertheless individual citizens' transversality conditions hold? What is a safe asset? What are its features? Why is government debt a safe asset? When does one lose the safe asset status? Why is there debt valuation puzzle for governments of advanced countries like the US and Japan? How do we have to modify representative agent asset pricing and the Fiscal Theory of the Price Level (FTPL) equation?

This paper attempts to address these questions within a setting in which citizens face uninsurable idiosyncratic risks and hence save for precautionary reasons. Each citizen lives forever and adjusts his portfolio consisting of physical capital and the government bond. Idiosyncratic shocks that cannot be diversified away (as well as aggregate shocks) make capital risky. This makes government bonds attractive since they can be sold after an adverse shock. From an individual citizen's perspective it is this ability to retrade, which makes the government bond a desirable hedging instrument. His planned dynamic trading strategy generates a payoff stream that is a good hedge. This is the first of the two key characteristics of a safe asset, the *Good Friend Analogy*.[1] A safe asset is like a good friend, it is around; that is, it is (i) valuable and (ii) liquid when one needs it.[2] Since a safe asset generates this extra service flow in the form of self-insurance, it is attractive even at a lower real interest rate, r, its cash flow return.

The higher the idiosyncratic risk citizens face, the higher is the precautionary savings demand and the lower is the required real rate of cash flow return r. When the interest rate on the safe asset is below the growth rate of the economy, $r < g$, the government can run a sustainable Ponzi scheme. Pay off the maturing bonds with newly issued debt and issuing more for additional expenditures. In such a setting, the standard asset pricing carries a bubble term and the government can "mine the bubble." Nevertheless citizens are willing to hold the bubbly safe asset given its service flow and their individual transversality conditions are satisfied.

[1]The two key characteristics of a safe asset were first proposed in Brunnermeier and Haddad (2012).

[2]Hence, it makes sense for central banks to act as market maker of last resort to ensure that bid-ask spreads remain low. Viewed this way John Law's big achievement was to create a safe asset status for English and French government debt early in the 18^{th} century.

A simple example is a setting in which the primary deficit of a government is a fixed fraction of GDP. That is, cash flows all bondholders earn as a group is always negative and growth at a rate of g. Discounting the future cash flows at $r < g$ yields a present value of minus infinity. The bubble term in the asset pricing equation is plus infinity. Such a debt valuation equation - or FTPL equation - is not very useful and one has to rely on the other equilibrium conditions to obtain the value of the government debt. Is there a different discount rate that is more economically meaningful and resurrects the power of the asset pricing (FTPL) equation? We show that r^{**} has these desired features. r^{**} is the risk-free rate excluding the component that is due to precautionary demand driven by the exposure to uninsurable idiosyncratic risk. r^{**} still reflects the time-preference rate, expected consumption growth rate as well as precautionary demand due to aggregate risk but not due to uninsurable idiosyncratic risk. We also show that r^{**} exceeds g and can be viewed as a "representative agent interest rate".

Using r^{**} as discount rate rate, we have to complement the discounted cash flow stream with a discounted stream of service flows - instead of a bubble term. Both terms are always finite, as $r^{**} > g$. Importantly, both are economically meaningful as they nicely separate the two benefits of the safe asset: cash flows, possibly negative, and a service flow that results from the ability to retrade. In other words, the real value of government debt, i.e. the nominal value $\mathcal{B}$ divided by the price level $\mathcal{P}$, is

$$\mathcal{B}/\mathcal{P} = \mathbb{E}[PV_{r^{**}}[primary\ surpluses]] + \mathbb{E}[PV_{r^{**}}[service\ flows]].$$

When adding aggregate shocks the full feature of safe assets emerges. We consider economies when entering a recession, aggregate output declines and at the same time idiosyncratic risk rises. The first term reflects the mainstream view, prevalent in the representative agent asset pricing. A drop in output reduces payoffs and increases the marginal utility, leading to the traditional positive β in the asset pricing equation. The second term, the service flow term, behaves very differently. As idiosyncratic risk rises in recessions, citizens prefer to shift their portfolio away from capital towards the government bond, resulting in a force that pushes up the real value of government debts. That is, the second term due to the discounted stream of service flows has a negative β. In a sense, the Jiang et al. (2019)'s "debt valuation puzzle" for the US can be seen as an empirical vindication of the importance of the second term in our analysis. Even more pronounced the primary surplus in Japan was negative for more than 50 out of the last 60 years, also suggesting a large second term overpowering the first term. Moreover, the issuer of a safe asset enjoys an exorbitant privilege which goes beyond the traditional convenience yield perspective as e.g. emphasized in Krishnamurthy and Vissing-Jorgensen (2012) since the presence of the safe asset affects not only the risk-free rate but also the yield of certain private bonds.

The second characteristic feature of safe assets is the *Safe Asset Tautology*. A safe asset is safe when it is perceived to be safe so that in times of crisis investors flock to it. In other words, the safe asset status is highly endogenous and part of a multiple equilibrium structure. From an aggregate perspective a safe asset is a "bubble" and bubbles can pop. As a consequence, an asset can lose its safe asset status. An explicit and formal characterization of the fragility of the safe asset status goes beyond the scope of this paper. In Brunnermeier et al. (2021) we discuss this in the context of an international framework for emerging market economies. Emerging market government bonds' safe asset status competes with advanced economies safe assets and hence are deeply affected by spillovers from US monetary policy.

Our safe asset perspective sheds not only a different light on the valuation of government debt but has also important implications debt sustainability analysis (DSA). First, as the government issues bonds at a faster pace citizens' cash-flow return of holding the government bonds declines. "Printing" bonds at a faster rate acts like a tax on bond holdings or, better said, on partial self-insurance through holding and retrading the safe asset. Increasing the tax rate increases the tax revenue, but erodes the "tax base", the value of the bonds. A "*Debt Laffer Curve*" emerges. When tax exceeds a certain level overall tax revenue from bubble mining declines. Second, any DSA should take the fragility of the safe asset status into account. Government debt is special as long as the government has sufficient fiscal space to fend off a possible jump to an non-safe asset equilibrium. Note that the ability alone to permanently raise taxes to back the debt is sufficient to prevent such a jump. This ability should be an important element in any debt sustainability analysis. Private companies do not have taxing power and hence can not fully replicate the off-equilibrium backing.

Our model has also interesting stock market asset pricing implications due to "flight-to safety" phenomena. During recessions idiosyncratic risk is assumed to rise. While for outside equity idiosyncratic risk can be diversified away, each household who manages her firm is exposed to her idiosyncratic risk via her inside equity holding. Hence, each citizen demands a higher risk premium during recession which depresses her demand for the (outside) equity stock index in favor of demand for the safe asset.

Literature. This paper touches upon many strands of classic and recent economic literature. We follow the safe asset definition outlined in Brunnermeier and Haddad (2012). Dang et al. (2015) emphasize the information insensitivity of safe assets. In Gorton and Pennachi (1990), Dang et al. (2017), and Greenwood et al. (2016) intermediaries create information insensitive assets. In He et al. (2019) model safe asset tautology within a generalized global games setting. Caballero et al. (2017) stress the importance of safe asset shortage. Brunnermeier et al. (2016, 2017) proposes a safe asset via securitization and argues that the main problem is the asymmetric supply of safe assets leading to eruptive cross-border capital flows.

This paper resolves the "Public Debt Valuation Puzzle" proposed in Jiang et al. (2019), which argues that the value of government debt should be significantly lowers not least because since primary surpluses, the total payments to all bond holders, are procyclical. In our setting the price of debt is procyclical since the bubble-term rises in bad times, resulting in a negative β asset. Second, it also resolves the "Government Debt Risk Premium Puzzle" (Jiang et al., 2020), the puzzle that the government debt appears to insure simultaneously bond holders and taxpayers whereas in standard models it can insure only one of the two groups. Our analysis shows that the bubble term can make the bond a negative β-asset, a good hedge for bond holders, while primary surpluses are countercyclical at the same time, thus providing insurance for taxpayers. Surprisingly, traded equity in our model exhibits excess volatility and predictability.

The value of government debt is inherently linked with fiscal debt sustainability. In deterministic models debt is sustainable and a Ponzi scheme is feasible if the risk free interest rate r is lower than the economic growth rate g. Bohn (1995) questions the simple r vs. g comparison for economies with aggregate risk. Overlapping Generations (OLG) include Samuelson (1958), Diamond (1965) with capital, Tirole (1985) with a bubble and most recently by Blanchard (2019). Models in which the risk-free rate is depressed due to uninsurable idiosyncratic risk go back to Bewley (1980). Aiyagari and McGrattan (1998) calibrates the optimal debt level in a Aiyagari-type model without aggregate risk. In these models no bubble can exist.[3] Angeletos (2007) studies idiosyncratic investment risks. In Brunnermeier and Sannikov (2016b,a) include a 'bubbly' safe asset in the form of government debt or money and allow for aggregate

[3]In Aiyagari models the risk-free rate and return on capital is higher than the zero growth rate of the economy.

risk. Bassetto and Cui (2018) and Brunnermeier et al. (2020a) expand the FTPL equation with a bubble. Reis (2020) also studies fiscal debt capacity for economies in which $r < g$, but the marginal return of capital $m > g$. Reis (2020) derives the level of bubble mining at which debt value is zero, i.e. when Debt Laffer Curve crosses zero. In Di Tella (2020) money yields additional utility. In Kiyotaki and Moore (2008) citizens self-insure against investment opportunity shocks. There is an extensive literature on rational bubbles. Survey papers include Miao (2014) and Martin and Ventura (2018).

2 Model

2.1 Model Setup

There is a continuum of households indexed by $i \in [0,1]$. All households have identical logarithmic preferences

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right]$$

with discount rate ρ.[4]

Each agent operates one firm that produces an output flow $a_t k_t^i dt$, where k_t^i is the capital input chosen by the firm and a_t is an exogenous productivity process that is common for all agents. Capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right) dt + \tilde{\sigma}_t d\tilde{Z}_t^i + d\Delta_t^{k,i},$$

where $d\Delta_t^{k,i}$ represents firm i's market transactions in physical capital, $\iota_t^i k_t^i dt$ are the firm's physical investment expenditures (in output goods), Φ is a concave function that captures adjustment costs in capital accumulation, δ is the depreciation rate, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across agents i. $\tilde{Z}^i$ introduces firm-specific idiosyncratic risk. $\tilde{\sigma}_t$ is an exogenous process that governs the magnitude of idiosyncratic risk faced by agents. To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log(1 + \phi\iota)$ with adjustment cost parameter $\phi \geq 0$ for the

[4] In Appendix **??**, we present a generalization with Duffie and Epstein (1992) preferences (continuous time Epstein and Zin (1989) preferences).

investment technology.

Each agent i can sell off some of the risky cash flows generated by capital k^i to capital markets as outside equity. Outside equity claims on i's capital have the same aggregate and idiosyncratic risk as capital itself, but may pay a lower expected return, reflecting an insider premium that i earns for managing the capital stock. Agents can hold a diversified equity portfolio and thereby eliminate idiosyncratic risk.

The key friction in the model is that agents are unable to share idiosyncratic risk perfectly. Specifically, we assume that agents face a skin-in-the-game constraint and must retain at least a fraction $\bar{\chi} \in (0, 1]$ of their capital in undiversified form, i.e. they can sell off at most a fraction $1 - \bar{\chi}$ of the cash flows generated by capital k^i as outside equity. As a consequence, agents have to bear the residual idiosyncratic risk $\bar{\chi}\tilde{\sigma}_t d\tilde{Z}^i$ inherent in their physical capital holdings.

Besides this limit on idiosyncratic risk sharing, there are no further financial frictions. Agents are allowed to trade physical capital and any type of claim contingent on aggregate risk.

In addition to households, there is a government that funds government spending, imposes taxes on firms, and issues nominal government bonds. The government has an exogenous need for real spending $\mathfrak{g}_t K_t dt$, where K_t is the aggregate capital stock and $\mathfrak{g}_t$ is an exogenous process. The government imposes a proportional output tax (subsidy, if negative) τ_t on firms. Outstanding nominal government debt has a face value of $\mathcal{B}_t$ and pays nominal interest i_t. $\mathcal{B}_t$ follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$, where the growth rate $\mu_t^{\mathcal{B}}$ is a policy choice of the government. In short, the government chooses the policy instruments τ_t, i_t, $\mu_t^{\mathcal{B}}$ contingent on histories of prices taking $\mathfrak{g}_t$ as given and subject to the nominal budget constraint

$$i_t \mathcal{B}_t + \mathcal{P}_t \mathfrak{g}_t K_t = \mu_t^{\mathcal{B}} \mathcal{B}_t + \mathcal{P}_t \tau_t a_t K_t, \tag{1}$$

where $\mathcal{P}_t$ denotes the price level.

We assume that the exogenous processes a_t, $\tilde{\sigma}_t$, $\mathfrak{g}_t$ follow a joint Markov diffusion process that is driven by some Brownian motion Z_t, which captures aggregate risk and is independent of all the idiosyncratic Brownian motions $\tilde{Z}_t^i$.

The model is closed by the aggregate resource constraint

$$C_t + \mathfrak{g}_t K_t + \iota_t K_t = a_t K_t, \tag{2}$$

where $C_t := \int c_t^i di$ is aggregate consumption and $\iota_t = \int \iota_t^i k_t^i / K_t di$ is the average investment rate.

2.2 Model Solution

Price Processes and Returns. Let q_t^K be the market price of a single unit of physical capital. Then, $q_t^K K_t$ is private capital wealth. Let further $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$ be the ratio of the real value of government debt to total capital in the economy.[5] Then, the real value of the total stock of government bonds is $q_t^B K_t$ and the real value of a single government bond is $\frac{q_t^B K_t}{\mathcal{B}_t}$. It is convenient to define the share of total wealth in the economy that is due to bond wealth,

$$\vartheta_t := \frac{q_t^B K_t}{\left(q_t^B + q_t^K\right) K_t}.$$

We postulate that q_t^B and q_t^K have a generic Ito evolution

$$dq_t^B = \mu_t^{q,B} q_t^B dt + \sigma_t^{q,B} q_t^B dZ_t, \qquad dq_t^K = \mu_t^{q,K} q_t^K dt + \sigma_t^{q,K} q_t^K dZ_t.$$

Whenever $q_t^B, q_t^K \neq 0$, the unknown (geometric) drifts $\mu_t^{q,B}, \mu_t^{q,K}$ and volatilities $\sigma_t^{q,B}, \sigma_t^{q,K}$ are uniquely determined by the local behavior of q_t^B and q_t^K, respectively. In the following, we also use the notation μ_t^ϑ and σ_t^ϑ for the (geometric) drift and volatility of ϑ_t.[6]

Households can trade two assets in positive net supply (if $q_t^B \neq 0$), bonds and capital. Assume that in equilibrium $\iota_t = \iota_t^i$ for all i (to be verified below) such that aggregate capital grows locally deterministically at rate $\Phi(\iota_t) - \delta$. Then, the return on bonds is

$$dr_t^{\mathcal{B}} = i_t dt + \frac{d\left(q_t^B K_t / \mathcal{B}_t\right)}{q_t^B K_t / \mathcal{B}_t} = \frac{d\left(q_t^B K_t\right)}{q_t^B K_t} - \overbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}^{=:\check{\mu}_t^{\mathcal{B}}} dt$$

[5] It is more convenient to work with this normalized version of the inverse price level $1/\mathcal{P}_t$, because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas q_t^B does not.

[6] This means, $d\vartheta_t = \mu_t^\vartheta \vartheta_t dt + \sigma_t^\vartheta \vartheta_t dZ_t$.

$$= \left(\Phi(\iota_t) - \delta + \mu_t^{q,B} - \breve{\mu}_t^{\mathcal{B}}\right) dt + \sigma_t^{q,B} dZ_t. \tag{3}$$

The return on agent i's capital is

$$\begin{aligned} dr_t^{K,i}\left(\iota_t^i\right) &= \frac{(1-\tau_t)\, a_t - \iota_t^i}{q_t^K} + \frac{d(q_t^K k_t^i)}{q_t^K k_t^i} \\ &= \left(\frac{(1-\tau_t)\, a_t - \iota_t^i}{q_t^K} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i. \end{aligned}$$

Using the government budget constraint (1) to substitute out $\tau_t a$ yields

$$dr_t^{K,i}\left(\iota_t^i\right) = \left(\frac{a_t - \mathfrak{g}_t - \iota_t^i}{q_t^K} + \frac{q_t^B}{q_t^K}\breve{\mu}_t^{\mathcal{B}} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma} d\tilde{Z}_t^i.$$

Outside equity claims issued by household i have the same risk characteristics as the capital return $dr_t^{K,i}$ but may have a different expected return. The return on outside equity issued by agent i is therefore

$$dr_t^{E,i} = \mathbb{E}_t[dr_t^{E,i}] + \sigma_t^{q,K} dZ_t + \tilde{\sigma} d\tilde{Z}_t^i,$$

where the expected return component $\mathbb{E}_t[dr_t^{E,i}]$ is determined in equilibrium. In equilibrium, all agents optimally hold a perfectly diversified equity portfolio. The return on that portfolio is

$$d\bar{r}_t^E = \int_0^1 r_t^{E,i} di = \mathbb{E}_t[d\bar{r}_t^E] + \sigma_t^{q,K} dZ_t.$$

Because all individual varieties of outside equity $dr_t^{E,i}$ generate the same aggregate risk contribution to the overall equity portfolio, it will be the case in equilibrium that $\mathbb{E}_t[dr_t^{E,i}] = \mathbb{E}_t[d\bar{r}_t^E]$ for all i.

Household Problem and Equilibrium. We formulate the household problem as a standard consumption-portfolio-choice problem that does not make explicit reference to the capital trading process $d\Delta_t^{k,i}$ as a choice variable. For this purpose, denote by n_t^i the net worth of household i and let $\theta_t^{k,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ be the fraction of net worth invested into capital, own outside equity, and the diversified portfolio of equity, respectively.[7]

[7] The own outside equity share $\theta_t^{E,i}$ is negative as this asset is issued by the household.

Then net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + dr_t^{\mathcal{B}} + \theta_t^{K,i}\left(dr_t^{K,i}\left(\iota_t^i\right) - dr_t^{\mathcal{B}}\right) + \theta_t^{E,i}\left(dr_t^{E,i} - dr_t^{\mathcal{B}}\right) + \theta_t^{\bar{E},i}\left(d\bar{r}_t^{E,i} - dr_t^{\mathcal{B}}\right). \tag{4}$$

The household chooses consumption c_t^i, real investment ι_t^i, and the portfolio shares $\theta_t^{k,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ in capital, own outside equity and the diversified equity portfolio, respectively, to maximize utility V_0^i subject to (4) and the skin-in-the-game constraint

$$-\theta_t^{E,i} \le (1-\bar{\chi})\theta_t^{K,i}. \tag{5}$$

The HJB equation for this problem is (using the returns expressions from the previous paragraph)

$$\begin{aligned}
&\rho V_t\left(n^i\right) - \partial_t V_t\left(n^i\right)\\
&= \max_{c^i,\theta^{K,i},\theta^{E,i},\theta^{\bar{E},i},\iota^i}\Bigg\{\log c^i + V_t'\left(n^i\right)\Bigg[-c^i + n^i\Bigg(\frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} + \theta^{K,i}\left(\frac{\mathbb{E}_t\left[d\hat{r}_t^{K,i}(\iota_t^i)\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt}\right)\\
&\qquad + \theta^{E,i}\left(\frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt}\right) + \theta^{\bar{E},i}\left(\frac{\mathbb{E}_t\left[d\bar{r}_t^{E}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt}\right)\Bigg)\Bigg]\\
&\qquad + \frac{1}{2}V_t''\left(n^i\right)\left(n^i\right)^2\left(\left(\sigma_t^{q,B} - \left(\theta^{K,i} + \theta^{E,i} + \theta^{\bar{E},i}\right)\frac{\sigma_t^{\vartheta}}{1-\vartheta_t}\right)^2 + \left(\theta^{K,i} + \theta^{E,i}\right)^2\tilde{\sigma}_t^2\right)\Bigg\},
\end{aligned}$$

where we have used $\sigma_t^{q,K} - \sigma_t^{q,B} = \frac{\sigma_t^{\vartheta}}{1-\vartheta_t}$. As this is a standard portfolio choice problem, we conjecture a functional form $V_t\left(n^i\right) = \alpha_t + \frac{1}{\rho}\log n^i$ for the value function,[8] where α_t depends on (aggregate) investment opportunities, but not on individual net worth n^i.

Substituting this into the HJB and taking first-order conditions with respect to c^i and ι^i yields the two equations

$$c_t^i = \rho n_t^i,$$
$$\frac{d}{d\iota}\frac{\mathbb{E}_t\left[d\hat{r}_t^{K,i}(\iota)\right]}{dt}\Bigg|_{\iota=\iota_t^i} = 0.$$

[8]The verification argument is entirely standard, see e.g. Brunnermeier et al. (2020a), Appendix A.2 for a proof.

The first condition is the familiar permanent income consumption equation for log preferences, the second condition reduces to a standard Tobin's q condition when combining it with the explicit formula for $d\hat{r}_t^{K,i}(\iota)$ given above:[9]

$$q_t^K = \frac{1}{\Phi'\left(\iota_t^i\right)}.$$

Using the functional form $\Phi(\iota) = \frac{1}{\phi}\log\left(1+\phi\iota\right)$ and goods market clearing (2), the first two equations aggregated across agents imply

$$\begin{aligned}
\iota_t &= \frac{(1-\vartheta_t)(a_t - \mathfrak{g}_t) - \rho}{1-\vartheta_t+\phi\rho}, \\
q_t^B &= \vartheta_t \frac{1+\phi(a_t-\mathfrak{g}_t)}{1-\vartheta_t+\phi\rho}, \\
q_t^K &= (1-\vartheta_t)\frac{1+\phi(a_t-\mathfrak{g}_t)}{1-\vartheta_t+\phi\rho},
\end{aligned}$$

which determines the equilibrium uniquely up to the nominal wealth share ϑ_t.

ϑ_t, in turn, is determined by agents' portfolio choice. Taking the first-order condition in the HJB with respect to the three portfolio shares $\theta_t^{k,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ yields three Merton portfolio choice equations

$$\begin{aligned}
\frac{\mathbb{E}_t\left[dr_t^{K,i}(\iota_t^i)\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= \left(\sigma_t^{q,B} - \frac{\theta_t^{K,i}+\theta_t^{E,i}+\theta_t^{\bar{E},i}}{1-\vartheta_t}\sigma_t^{\vartheta}\right)\frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \left(\theta_t^{K,i}+\theta_t^{E,i}\right)\tilde{\sigma}_t^2 - \lambda_t^i\left(1-\bar{\chi}\right), \\
\frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= \left(\sigma_t^{q,B} - \frac{\theta_t^{K,i}+\theta_t^{E,i}+\theta_t^{\bar{E},i}}{1-\vartheta_t}\sigma_t^{\vartheta}\right)\frac{\sigma_t^{\vartheta}}{1-\vartheta_t} + \left(\theta_t^{K,i}+\theta_t^{E,i}\right)\tilde{\sigma}_t^2 - \lambda_t^i, \\
\frac{\mathbb{E}_t\left[d\bar{r}_t^{E}\right]}{dt} - \frac{\mathbb{E}_t\left[dr_t^{\mathcal{B}}\right]}{dt} &= \left(\sigma_t^{q,B} - \frac{\theta_t^{K,i}+\theta_t^{E,i}+\theta_t^{\bar{E},i}}{1-\vartheta_t}\sigma_t^{\vartheta}\right)\frac{\sigma_t^{\vartheta}}{1-\vartheta_t}.
\end{aligned}$$

Here, λ_t^i denotes the Lagrange multiplier on the constraint (5). Combining the last two equations and using $\frac{\mathbb{E}_t\left[d\bar{r}_t^E\right]}{dt} = \frac{\mathbb{E}_t\left[dr_t^{E,i}\right]}{dt}$ in equilibrium, we obtain a simple characterization of λ_t^i:

$$\lambda_t^i = \left(\theta_t^{K,i} + \theta_t^{OE,i}\right)\tilde{\sigma}_t^2.$$

[9] In particular, this equation implies that $\iota_t^i = \iota_t$.

In particular, λ_t^i is always positive and thus the constraint (5) must be binding – households issue the maximum possible amount of outside equity. Then, $\theta_t^{K,i} + \theta_t^{E,i} = \theta_t^{K,i}\bar{\chi}$. Substituting this, the expression for λ_t^i and the return expressions into the first of the three portfolio choice conditions yields

$$\frac{a_t - \mathfrak{g}_t - \iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \breve{\mu}_t^\mathcal{B}}{1-\vartheta_t} + \frac{\left(\sigma_t^{q,B} - \sigma_t^\vartheta\right)\sigma_t^\vartheta}{1-\vartheta_t} = \left(\sigma_t^{q,B} - \frac{\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i}}{1-\vartheta_t}\sigma_t^\vartheta\right)\frac{\sigma_t^\vartheta}{1-\vartheta_t} + \theta_t^{K,i}\bar{\chi}^2\tilde{\sigma}_t^2.$$

The fact that all households choose the same portfolio shares and equity market clearing immediately imply $\theta_t^{E,i} = -\theta_t^{\bar{E},i}$. Furthermore, bond market clearing then requires $1 - \theta_t^{K,i} = \vartheta_t$. Substituting these clearing conditions and goods market clearing into the previous equation and solving for μ_t^ϑ gives us a condition for ϑ_t:

$$\mu_t^\vartheta = \rho + \breve{\mu}_t^\mathcal{B} - (1-\vartheta_t)^2\bar{\chi}^2\tilde{\sigma}_t^2.$$

This is a backward equation for ϑ_t that has been derived under the assumption that bonds have a positive value ($\vartheta_t > 0$). In particular, in these cases multiplying the equation by ϑ_t represents an equivalence transformation. Furthermore, if $\vartheta_t = 0$, then by no arbitrage, agents must expect also $d\vartheta_t = 0$; otherwise, they could earn an infinite risk-free return from investing into bonds. Consequently, the backward stochastic differential equation (BSDE)

$$\mathbb{E}_t\left[d\vartheta_t\right] = \left(\rho + \breve{\mu}_t^\mathcal{B} - (1-\vartheta_t)^2\bar{\chi}^2\tilde{\sigma}_t^2\right)\vartheta_t dt \tag{6}$$

must hold along any equilibrium path, regardless of whether bonds have positive value or not.

Together with a specification for the evolution of the exogenous states $\tilde{\sigma}_t$, a_t, and $\mathfrak{g}_t$ and for policy $\breve{\mu}_t^\mathcal{B}$, equation (6) determines the equilibrium process for ϑ_t.

2.3 Safe Asset Debt Valuation Equation: Two Perspectives

The value of government debt has to satisfy a debt valuation equation that relates the real value of debt to the present value of future primary surpluses. There are two ways to derive such an equation: (1) by iterating the government's flow budget constraint forward in time and pricing the total stock of government bonds with any marginal agent's stochastic discount factor (SDF) or (2) by aggregating individual households' intertemporal budget constraint and imposing the aggregate resource constraint. Both procedures imply the same valuation equation with complete markets, but, with incomplete markets, lead to two distinct equations that differ in the effective discount rate applied to government surpluses. These equations provide two different perspectives for pricing government debt.

The first procedure leads to a "buy and hold perspective" of government debt pricing. The value of government debt must equal the marginal valuation of an individual agent that buys and holds a (small) constant fraction of the total stock of outstanding bonds.[10] The cash flow stream associated with this strategy equals precisely the stream of primary surpluses. Hence, in a setting without aggregate risk the bond is risk-free and future payoffs are discounted at the risk-free rate. In a setting with aggregate risk, only the aggregate component of the stochastic discount factor enters the debt valuation equation.

The second procedure leads to a "dynamic trading perspective" of government debt pricing. It recognizes that individual citizens do not intend to buy and hold the government bond, but plan to retrade it whenever they face a shock. After a negative shock, they raise cash flow by selling the bond, while after a positive shock they buy additional bonds. The cash flow stream associated with this optimal trading strategy is (idiosyncratically) stochastic. The procedure effectively prices these cash flows from the optimal stochastic trading strategy of individuals and then aggregate in a second step. The resulting equation contains a "service flow" term from retrading that is absent in the buy and hold perspective.

Note also dynamic programming implies that the transversality condition has to hold only from the dynamic trading perspective, for each individual agent. Optimality does not imply a transversality condition from the buy and hold perspective (where discounting happens at a lower effective rate). For that reason a gap between the value of debt and the present value of surpluses may appear from the buy and hold perspective that is closed by an additional bubble term.

[10]This may require the agent to trade despite the label "buy and hold", but only directly with the issuer, the government, in order to absorb new debt issuance, not with other agents.

Buy and Hold Perspective. We denote the individual SDF process of citizen i with ξ^i_t. This process satisfies $\xi^i_0 = 1$ and $d\xi^i_t/\xi^i_t = -r^f_t dt - \varsigma_t dZ_t - \tilde{\varsigma}^i_t d\tilde{Z}^i_t$, with a negative drift term equal to the risk-free rate and aggregate and idiosyncratic price of risk terms, ς_t, $\tilde{\varsigma}^i_t$ respectively.[11] From the buy and hold perspective, individual uninsurable risk does not enter the valuation equation directly, so that only the aggregate component $\bar{\xi}_t$ of the processes ξ^i_t matters, i.e. $d\bar{\xi}_t/\bar{\xi}_t = -r^f_t dt - \varsigma_t dZ_t$.[12] Absent aggregate shocks (including inflation shocks), the government bond is a risk-free asset and the relevant discount factor is simply $\bar{\xi}_t = \exp(-\int_0^t r^f_\tau d\tau)$.

The government debt valuation equation from the buy and hold perspective at $t = 0$ is

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty} \left(\mathbb{E}\left[\int_0^T \bar{\xi}_t s_t K_t dt \right] + \mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right). \tag{7}$$

This equation consists of two terms: a discounted stream of primary surpluses plus (the limit of) a discounted terminal value. The latter can be positive even in the limit, giving rise to a possible bubble on government debt.[13] The reason is that in our model no private citizen's transversality condition necessary implies $\mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T}\right] \to 0$ because agents do not buy and hold a fixed fraction of the government debt stock but constantly trade bonds. If the discount factor is small enough so that the terminal condition does converge to zero, we obtain the traditional debt valuation equation that says that the value of debt must equal the present value of primary surpluses.

To obtain equation (7), we start by using $d\mathcal{B}_t = \mu^{\mathcal{B}}_t \mathcal{B}_t dt$ to rewrite the government

[11] In integral form the individual SDF is

$$\xi^i_t = \underbrace{\exp\left(-\int_0^t r^f_\tau d\tau\right)}_{\text{time discounting}} \cdot \underbrace{\exp\left(-\int_0^t \varsigma_t dZ_\tau - \frac{1}{2}\int_0^t \varsigma^2_\tau d\tau\right)}_{\text{aggregate risk}} \cdot \underbrace{\exp\left(-\int_0^t \tilde{\varsigma}_\tau d\tilde{Z}^i_\tau - \frac{1}{2}\int_0^t \tilde{\varsigma}^2_\tau d\tau\right)}_{\text{idiosyncratic risk}},$$

where the second and third factors are martingales.

[12] The aggregate discount factor is the projection of any individual citizen's SDF onto a common filtration generated by the aggregate Brownian $\{Z_t\}_{t=0}^\infty$. Put differently, $\bar{\xi}_t := \mathbb{E}\left[\xi^i_t \mid Z_\tau : \tau \le t\right]$, takes conditional expectations with respect to the history of aggregate shocks dZ_τ up to time t but without any knowledge of idiosyncratic shocks. Equivalently, $\bar{\xi}_t = \int \xi^i_t di$ is the unweighted average of individual SDFs.

[13] The bubble term on government debt is discussed in detail in Brunnermeier et al. (2020a).

flow budget constraint (1) as

$$-\left(d\mathcal{B}_t - i_t\mathcal{B}_t dt\right) = \mathcal{P}_t \underbrace{\left(\tau a_t - \mathfrak{g}_t\right)}_{=s_t} K_t dt,$$

where s_t denotes again the government primary surplus normalized by the aggregate capital stock.

We now multiply both sides by the nominal SDF $\xi_t^i/\mathcal{P}_t$ of agent i and use Ito's product rule to replace $\xi_t^i/\mathcal{P}_t d\mathcal{B}_t$ with $d\left(\xi_t^i/\mathcal{P}_t\mathcal{B}_t\right) - \mathcal{B}_t d(\xi_t^i/\mathcal{P}_t)$:[14]

$$-d\left(\xi_t^i\mathcal{B}_t/\mathcal{P}_t\right) + \mathcal{B}_t\left(d\left(\xi_t^i/\mathcal{P}_t\right) + i_t\xi_t^i/\mathcal{P}_t dt\right) = \xi_t^i s_t K_t dt.$$

Integrating this equation from $t = 0$ to $t = T$, taking expectations, and solving for $\xi_0^i\mathcal{B}_0/\mathcal{P}_0$ yields

$$\xi_0^i\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^T \xi_t^i s_t K_t dt\right] - \mathbb{E}\left[\int_0^T \mathcal{B}_t\left(d\left(\xi_t^i/\mathcal{P}_t\right) + i_t\xi_t^i/\mathcal{P}_t dt\right)\right] + \mathbb{E}\left[\xi_T^i\frac{\mathcal{B}_T}{\mathcal{P}_T}\right]. \tag{8}$$

Equation (8) is simply an accounting identity, the government flow budget constraint (1) multiplied with the discounting process $\xi_t^i/\mathcal{P}_t$. We now add economic content by noting that the individual SDF ξ_t^i must price the bond because agent i is marginal in the bond market. This implies that the associated nominal SDF $\xi_t^i/\mathcal{P}_t$ must decay on average at the nominal market interest rate, so that the second term in equation (8) vanishes. In addition, we can replace the individual SDF ξ_t^i with the average SDF $\bar{\xi}_t$ because equation (8) holds for all individuals i and s_tK_t and $\mathcal{B}_T/\mathcal{P}_T$ are free of idiosyncratic risk. When taking the limit $T \to \infty$, we obtain equation (7)

Dynamic Trading Perspective. Let $\eta_t^i := n_t^i/N_t$ be citizen i's net worth share and denote again i's SDF process by ξ_t^i. Pricing individual bond portfolios and aggregating over agents i yields our main valuation equation from the dynamic trading perspective,

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \int\left(\mathbb{E}\left[\int_0^\infty \xi_t^i\cdot\eta_t^i s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i\cdot\eta_t^i\left(1-\vartheta_t\right)^2\tilde{\chi}^2\tilde{\sigma}_t^2\frac{\mathcal{B}_t}{\mathcal{P}_t}dt\right]\right)di. \tag{9}$$

[14]There is no quadratic covariation term because $d\mathcal{B}_t$ is absolutely continuous.

The real value of all outstanding public debt $\mathcal{B}_0/\mathcal{P}_0$ is the integral of the valuations of individual debt holdings. Each of these valuations consists of two terms, the discounted value of the share of future primary surpluses, $\eta_t^i s_t K_t := \eta_t^i(\tau_t a - \mathfrak{g}_t)K_t$, paid out to agent i plus the discounted value of future service flows, $\eta_t^i (1-\vartheta_t)^2 \bar{\chi}^2\tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t}$, that agent i derives from trading bonds. The safe asset service flow is due to partial insurance, which increases in the value of public debt, and the amount of idiosyncratic risk the citizen is exposed to, which in turn depends on his portfolio share on physical capital $(1-\vartheta_t)$ and undiversified risk $\bar{\chi}\tilde{\sigma}_t$. Government bonds provide a positive service flow because the agent sells bonds precisely when she experiences a negative idiosyncratic shock, so that the bond portfolio generates a positive payout precisely in times of high marginal utility ξ_t^i.

Equation (9) emphasizes that the total value is obtained by aggregating individual portfolio valuations. Mathematically, it is more convenient to interchange the order of integration and write the equation as

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]. \quad (10)$$

This equation discounts aggregate cash flows (surpluses and service flows) free of idiosyncratic risk like equation (7) obtained from the buy and hold perspective. But importantly, the "stochastic discount factor" in this equation is a net-worth-weighted average of individual stochastic discount factors. Since a single citizen's individual net worth weight η_t^i co-moves negatively with his SDF ξ_t^i, the discount factor is lower (discount rate is higher) than the usual unweighted average discount factor (used in the buy and hold perspective above). It turns out this weighted average SDF is not a mere mathematical artifact from swapping integrals but has also an economic interpretation as the correct intertemporal ratio of marginal utilities of aggregate cash flows for a pseudo-representative agent who is forced to distribute aggregate consumption to individuals according to the equilibrium consumption shares c_t^i/C_t in our model. We discuss this interpretation in more detail below.

To obtain valuation equations (9) and (10), we start valuing citizen i's bond portfolio at time $t = 0$:[15]

$$n_0^{b,i} = \mathbb{E}\left[\int_0^\infty \tilde{\xi}_t^i \Big(\underbrace{c_t^i}_{\text{consumption}} - \underbrace{k_t^i\left(a - \iota_t - \tau_t a_t\right)}_{\text{production net of investment and taxes}} + \underbrace{q_t^K k_t^i \tilde{\sigma}_t^{\Delta,i}\left(-\tilde{\varsigma}_t^i\right)}_{\text{cash flows from trading capital}} \Big) dt\right], \tag{11}$$

where $n_0^{b,i} := \theta_0^i n_0^i$ is the initial bond wealth of agent i. This equation says that the initial bond wealth of the household must equal the discounted value of future payouts from the bond portfolio. Payouts consist of consumption in excess of the citizen i's own production net of reinvestment and tax payments and of trading expenses for purchasing new capital (these can be negative if capital is sold). In this model, capital trading only happens in response to idiosyncratic shocks, $d\Delta_t^{k,i} = \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$.[16] In the appendix we show that $\tilde{\sigma}_t^{\Delta,i} = -\vartheta_t \bar{\chi} \tilde{\sigma}_t$, that is agents sell capital and buy bonds when they receive a positive shock and vice versa, and that $\tilde{\varsigma}_t^i = (1 - \vartheta_t)\,\bar{\chi}\tilde{\sigma}_t$.

Next, replacing individual with scaled aggregate variables, $c_t^i = \eta_t^i C_t$ and $k_t^i = \eta_t^i K_t$, one obtains

$$n_0^{b,i} = \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i \left(\tau_t a_t K_t + C_t - (a - \iota_t)\,K_t + (1 - \vartheta_t)\,\bar{\chi}^2 \tilde{\sigma}_t^2 \vartheta_t q_t^K K_t\right) dt\right].$$

Including the aggregate resource constraint (2), $C_t - (a - \iota_t)\,K_t = \mathfrak{g}_t K_t$, the fact that, $n_0^{b,i} = \vartheta_0 n_0^i = \eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0}$, and, $\vartheta_t q_t^K K_t = (1 - \vartheta_t)\,\frac{\mathcal{B}_t}{\mathcal{P}_t}$ leads to

$$\eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \xi_t^i \eta_t^i \,(1 - \vartheta_t)^2\, \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]. \tag{12}$$

Finally, integrating over individuals i yields equation (9).

Comparison of the Two Approaches. The SDFs used in equations (7) and (10) are both free of idiosyncratic risk and imply the same aggregate risk premium, but they differ with respect to their average rate of decay, the "risk-free rate" they imply. While the average SDF $\bar{\xi}$ decays at rate r_t^f and is thus a "proper" SDF in this model that prices

[15]This equation is an immediate consequence of the agent's intertemporal budget constraint. In particular, a transversality condition always ensures that there is no additional nonvanishing terminal wealth term.

[16]While agents expect to make as many purchases as sales in the future, so that the expected cash flows from trading are zero, there is nevertheless a trading term in equation (11) that reflects the covariance between cash flows from trading ($q_t^K k_t^i \tilde{\sigma}_t^{\Delta,i}$) and individual marginal utility ($-\tilde{\varsigma}_t^i$).

all assets free of idiosyncratic risk, this is not true for the weighted average SDF $\int \xi_t^i \eta_t^i di$. The latter decays at a rate $r_t^f + \tilde{\varsigma}_t \tilde{\sigma}_t^n$, where $\tilde{\sigma}_t^n$ is the idiosyncratic net worth volatility of agents (which is identical for all agents in equilibrium). The weighted average SDF $\int \xi_t^i \eta_t^i di$ therefore discounts safe cash flows at a higher rate than the risk-free rate. The reason for this is apparent from equation (9) which inverts the order of integration: while aggregate cash flows from bonds are free of idiosyncratic risk, each agent holds a stochastic share η_t^i of the aggregate bond portfolio so that individual bond portfolios do contain idiosyncratic risk.

These considerations imply that only equation (7) is a standard asset pricing condition, a discounted present value formula using a SDF that prices all assets (at least those free of idiosyncratic risk). However, (7) can have a nonzero bubble term. Unfortunately, it can even happen that both the bubble term and the present value of primary surpluses are infinite with opposite sign, yet their sum still converges as $T \to \infty$. In contrast, the integrals in equation (10) are always well-defined and finite. Working with equation (10) can therefore be more informative, even though it uses a SDF that does not price the assets in this economy without additional service flow terms.

Relating the Dynamic Trading Perspective to a Representative Agent. The weighted-average SDF is not a "proper" SDF that prices assets in the competitive equilibrium of our incomplete markets economy, yet it turns out to be the correct SDF of a representative agent in a Lucas-type asset pricing economy that generates the same allocation as our competitive equilibrium. In addition, if we interpret aggregate capital and aggregate bonds as two "trees" in this representative agent economy, then equation (10) is precisely the valuation equation for the bond tree from the perspective of the representative agent. The dynamic trading perspective is therefore equivalent to the perspective of a hypothetical representative agent.

More precisely, consider a representative agent that maximizes a weighted welfare function $\mathcal{W}_0 = \int \lambda^i V_0^i di$ with some (positive) welfare weights $(\lambda^i)_{i\in[0,1]}$. If we denote by $\eta_t^i := c_t^i / C_t$ the consumption share of agent i, we can write utility of this representative agent as

$$\mathcal{W}_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \int \lambda^i \log\left(\eta_t^i C_t\right) di dt\right], \tag{13}$$

which resembles standard time-separable utility in *aggregate consumption* C_t with period utility function $C_t \mapsto \int \lambda^i u(\eta_t^i C_t) di$. The consumption shares η_t^i in this utility function evolve according to $d\eta_t^i = \tilde{\sigma}_t^\eta d\tilde{Z}_t^i$ with volatility process $\tilde{\sigma}_t^\eta$ specified below in equation (15). We show in Appendix X that $\mathcal{W}_0$ can also be written as

$$\mathcal{W}_0 = w_0 + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log C_t - \frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2\right) dt\right] \tag{14}$$

with some constant w_0. Equation (14) eliminates the direct dependence on i and gives us the alternative interpretation that two "goods" enter the representative agent's utility function, an aggregate consumption good and a "volatility reduction good" which is captured by the term $-\frac{1}{2\rho}\left(\tilde{\sigma}_t^\eta\right)^2$.[17]

We assume that the representative agent has access to two assets, capital K_t, which produces a certain bundle of the aggregate consumption good and volatility $\tilde{\sigma}_t^\eta$, and "derivatives" X_t, which mimic the cash flows to individuals i generated by bond trades in our incomplete markets model and thereby reduce volatility. Capital grows at rate $g_t := \Phi(\iota_t) - \delta$ over time and produces output net of reinvestment and taxes at rate $\big((1-\tau_t)a_t - \iota_t\big) K_t dt$. The face value X_t of derivatives evolves according to

$$dX_t/X_t = g_t dt + \sigma_t^{q,B} dZ_t$$

where $\sigma_t^{q,B}$ is the volatility process of q_t^B implied by the competitive equilibrium of the incomplete markets model. Derivatives generate a cash flow $-\breve{\mu}_t^{\mathcal{B}} X_t$ and reduce fluctuations in consumption shares η_t^i. Specifically, the volatility loading $\tilde{\sigma}_t^\eta$ satisfies the equation

$$\left(q_t^K K_t + X_t\right)\tilde{\sigma}_t^\eta = q_t^K K_t \bar{\chi} \tilde{\sigma}_t, \tag{15}$$

where q_t^K is the capital price process from the incomplete markets economy. We can interpret the product $X_t\tilde{\sigma}_t^\eta$ as the aggregate gross trading cash flows from bond trades in response to idiosyncratic shocks in the incomplete markets economy.[18]

Let Q_t^K be the capital price that the representative agent faces, P_t^X the price per unit (face value) of derivatives, and let $N_t := Q_t^K K_t + P_t^X X_t$ be the representative agent's

[17] The representative agent's objective is akin to a money in utility (MIU) model. Holding the derivative asset introduced below reduces volatility $\tilde{\sigma}_t^\eta$ in a similar way as holding money in a MIU model generates utility services.

[18] $q_t^K k_t^i \bar{\chi} \tilde{\sigma}_t$ is sensitivity of an agent i's capital wealth to shocks $d\tilde{Z}_t^i$ before portfolio rebalancing and $q_t^K k_t^i \tilde{\sigma}_t^\eta$ is the shock sensitivity after rebalancing. The difference, $q_t^K k_t^i\left(\bar{\chi}\tilde{\sigma}_t - \tilde{\sigma}_t^\eta\right)$ measures trading cash flows per unit of $d\tilde{Z}_t^i$ and aggregating over all agents yields $X_t\tilde{\sigma}_t^\eta$.

total net worth. The budget constraint of the representative agent is

$$dN_t = -C_t dt + Q_t^K K_t dr_t^K + P_t^X X_t dr_t^X \tag{16}$$

with return processes

$$dr_t^K = \left(\frac{(1-\tau_t)a_t - \iota_t}{Q_t^K} + \mu_t^{Q,K} + g_t \right) dt + \sigma_t^{Q,K} dZ_t,$$
$$dr_t^X = \left(\mu_t^{P,X} + g_t - \check{\mu}_t^{\mathcal{B}} + \sigma_t^{q,B}\sigma_t^{P,X} \right) dt + \left(\sigma_t^{q,B} + \sigma_t^{P,X} \right) dZ_t.$$

The representative agent maximizes utility $\mathcal{W}_0$ subject to the budget constraint (16) and the risk constraint (15) taking the prices Q_t^K, P_t^X and the return processes as given. The representative agent model is closed by time-zero supplies of capital (K_0) and derivatives (X_0). We impose the additional relationship $X_0 = q_0^B K_0$, where q_0^B is the initial value of q_t^B in the incomplete markets model. While this supply restriction for X_0 may appear ad hoc, it can be micro-founded in an environment with information frictions in which idiosyncratic shocks are private information and agents have access to hidden trade and savings.[19] In such an environment, incentive compatibility requires that any insurance transfer to an agent must be precisely offset by a reduction in the present value of that agent's future consumption. Otherwise, the agent would have incentives to misreport the size of the shock and secretly trade capital. Incentive compatibility thus limits the amount of insurance that can be provided, i.e. the quantity X of derivatives.

We show in Appendix X that the competitive equilibrium of this representative agent economy features prices $Q_t^K = q_t^K$ and $P_t^X = 1$ (and thus $P_t^X X_t = q_t^B K_t$), so that asset prices are the same as in the incomplete markets economy.[20] Using the utility representation (13), we see immediately that the representative agent's SDF process is

$$\Xi_t = e^{-\rho t} \frac{\int \lambda^i \eta_t^i u'(\eta_t^i C_t) di}{\int \lambda^i \eta_0^i u'(\eta_0^i C_0) di} = \frac{\int \lambda^i u'(c_0^i) \xi_t^i \eta_t^i di}{\int \lambda^i u'(c_0^i) di}$$

We show in the appendix that Ξ_t is independent of welfare weights λ^i and thus we can

[19]Details on this micro-foundation can be found in Brunnermeier et al. (2020b). This information environment has also been employed by Di Tella (2020) in a closely related model.

[20]Also aggregate consumption C_t and the consumption shares η_t^i are as in the incomplete markets economy. The representative agent economy therefore leads to the same allocation.

assume w.l.o.g. that $\lambda^i u'(c_0^i)$ is a constant independent of i.[21] This implies $\Xi_t = \int \xi_t^i \eta_t^i di$, *the representative agent's SDF equals the weighted-average SDF*. The valuation equation for derivatives from the perspective of the representative agent is

$$P_0^X X_0 = \mathbb{E}\left[\int_0^\infty \Xi_t \cdot \left(-\breve{\mu}_t^{\mathcal{B}} X_t\right) dt\right] + \mathbb{E}\left[\int_0^\infty \Xi_t \cdot (1-\vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 P_t^X X_t dt\right]. \tag{17}$$

Here, the first term represents the discounted present value of cash flows $-\breve{\mu}_t^{\mathcal{B}} X_t$ and the second term represents the discounted volatility reduction service flows that derivatives provide by lowering $\tilde{\sigma}^\eta$ in the utility function (14). As derivatives in the representative agent economy play the same role as bonds in the incomplete markets economy, we can make the identification $P_0^X X_t = X_t = q_t^B K_t$ and $-\breve{\mu}_t^{\mathcal{B}} X_t = s_t K_t$. With these replacements, equation (17) becomes equation (10), the debt valuation equation from the dynamic trading perspective.

2.4 Closed-Form Steady State and Gordon Growth Formulas

In this section, we assume that productivity a, idiosyncratic risk $\tilde{\sigma}$, and government spending per unit of capital $\mathfrak{g}$ are constant. We also restrict attention to government policies that hold taxes τ constant over time and characterize steady-state equilibria with constant q^B and q^K and a positive value of government bonds, $q^B > 0$. These assumptions immediately imply that also ϑ and $\breve{\mu}^{\mathcal{B}}$ must be constant in such a steady state.

Any such equilibrium must thus solve equation (6) with $d\vartheta_t = 0$. The right-hand side is a third-order polynomial, so there are three solutions to this equation, $\vartheta = 0$, $\vartheta = \frac{\bar{\chi}\tilde{\sigma} + \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\bar{\chi}\tilde{\sigma}}$, and $\vartheta = \frac{\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\bar{\chi}\tilde{\sigma}}$. Among these solutions, only the third can be consistent with $q^B, q^K > 0$ and thus a valid steady state equilibrium in which bonds have a positive value.[22] It is consistent with such an equilibrium if in addition the condition

$$\bar{\chi}\tilde{\sigma} \geq \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}$$

is satisfied. Effectively, this inequality imposes a constraint on bond growth in excess

[21] $\lambda^i u'(c_0^i)$ would also be independent of i if we allowed the representative agent to choose the initial consumption allocation.

[22] The second solution never corresponds to a valid equilibrium, while the first is only consistent with equilibrium if government primary surpluses are zero, see Brunnermeier et al. (2020a) for details.

of interest payments $\breve{\mu}^{\mathcal{B}}$ for the private sector to remain willing to hold government bonds.

In this case, investment is

$$\iota = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(a - \mathfrak{g}) - \rho\bar{\chi}\tilde{\sigma}}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}$$

and the (scaled) real asset values are

$$q^B = \frac{\left(\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}, \qquad q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}.$$

While these expressions have the advantage of being an explicit model solution in terms of parameters, for interpretation it is helpful to write the last two equations as Gordon growth formulas

$$q^B = \frac{s + (1-\vartheta)^2\bar{\chi}^2\tilde{\sigma}^2 q^B}{\mathbb{E}[dr^n]/dt - g}, \qquad q^K = \frac{(1-\tau)a - \iota}{\mathbb{E}[dr^K]/dt - g}.$$

Here, the second equation follows from the fact that the price of a single unit of capital must be the present value of cash flows generated by that unit of capital. The current period cash flow is production net of taxes and reinvestment, $(1-\tau)a - \iota$, and the expected growth rate of these cash flows is the economy's growth rate $g := \Phi(\iota) - \delta$.[23] Because capital is risky, expected cash flows must be discounted at the expected return on capital $\mathbb{E}[dr^K]/dt$, which includes a risk premium for idiosyncratic risk.

The first equation is a consequence of equation (10), the individual perspective on government debt valuation. Per unit of aggregate capital in the economy, the "cash flow" on bonds consists of the surplus-capital ratio s and the service flow $(1-\vartheta)^2\bar{\chi}\tilde{\sigma}^2 q^B$ from trading bonds to self-insure against idiosyncratic risk. Both types of cash flows grow on average at the economy's growth rate, but are risky from the individual's perspective. The required discount rate is therefore $\mathbb{E}[dr^n]/dt$, where $dr^n = \vartheta dr^{\mathcal{B}} + (1-\vartheta)dr^K$ denotes the return on the agents (net worth) portfolio, because the idiosyncratic risk of net worth is precisely the residual idiosyncratic risk that the agent has to bear

[23] g is both the growth rate of output and of the aggregate capital stock.

after optimal re-trading of bonds.[24]

3 Counter-cyclical Safe Asset and 2 Betas

3.1 Model Setup with Stochastic Idiosyncratic Risk and Recursive Utility

We introduce aggregate risk as shocks to idiosyncratic risk $\tilde{\sigma}_t$. We interpret periods of high idiosyncratic risk as recessions and want them to be associated with lower consumption and higher marginal utility. Rather than microfounding this relationship explicitly, we simply impose exogenous relationships $a_t = a(\tilde{\sigma}_t)$ and $\mathfrak{g} = \mathfrak{g}(\tilde{\sigma}_t)$ that are consistent with the desired correlation structure.[25]

For idiosyncratic risk $\tilde{\sigma}_t$, we specify a Heston (1993) model of stochastic volatility, i.e. we assume that the idiosyncratic variance $\tilde{\sigma}_t^2$ follows a Cox–Ingersoll–Ross process (Cox et al., 1985) process,

$$d\tilde{\sigma}_t^2 = -\psi\left(\tilde{\sigma}_t^2 - \left(\tilde{\sigma}^0\right)^2\right)dt - \sigma\tilde{\sigma}_t dZ_t$$

with parameters $\psi, \sigma, \tilde{\sigma}^0 > 0$.

To ensure that C_t/K_t is strictly decreasing in $\tilde{\sigma}_t$, we do not directly specify functions $a(\tilde{\sigma})$ and $\mathfrak{g}(\tilde{\sigma})$, but instead impose for the endogenous consumption-capital ratio the equation

$$C/K(\tilde{\sigma}_t) := \alpha_0 - \alpha_1\tilde{\sigma}_t$$

for some parameters $\alpha_0, \alpha_1 > 0$. Because equation (6) implies that ϑ_t is determined independently of the processes for a_t and $\mathfrak{g}_t$,[26] we can first solve for the solution function $\vartheta(\tilde{\sigma})$ using just the specification for the $\tilde{\sigma}_t$ process and then invert the formula $C_t/K_t = \rho\frac{1+\phi(a_t-\mathfrak{g}_t)}{1-\vartheta_t+\phi\rho}$ to back out the required function $a - \mathfrak{g}$ to obtain the desired

[24] More formally, $\eta_t^i = n_t^i/N_t$, so that the relative risk in n_t^i is the same as the relative risk in η_t^i and the latter matters for discounting when using the weighted-average discount factor $\int \xi_t^i \eta_t^i di$.

[25] For models similar to ours in which output and consumption naturally react negatively to risk shocks, see DiTella and Hall (2020) and Li and Merkel (2020).

[26] This is also true for the extension with stochastic differential utility introduced below.

consumption-capital ratio in equilibrium.[27]

For government policy, we assume that debt growth net of interest payments satisfies a linear relationship

$$\breve{\mu}_t^{\mathcal{B}} = -\nu_0 + \nu_1 \tilde{\sigma}_t \tag{18}$$

with parameters $\nu_0, \nu_1 > 0$. Provided ν_1 is sufficiently large, this implies that surpluses $s_t = -\breve{\mu}_t^{\mathcal{B}} q_t^B$ are positive for low idiosyncratic risk (in expansions) and negative for high idiosyncratic risk (in recessions). Primary surpluses therefore correlate negatively with marginal utility and any agent in the economy would require a positive risk premium for holding a (hypothetical) claim to primary surpluses.

Finally, in order for our model to generate quantitatively realistic aggregate risk premia, we work in this section with a slightly more general version of the model. Specifically, we replace logarithmic preferences of households with stochastic differential utility (Duffie and Epstein, 1992) with unit elasticity of intertemporal substitution and arbitrary relative risk aversion $\gamma > 0$: household i maximizes V_0^i, where V_t^i is recursively defined by

$$V_t^i = \mathbb{E}_t \left[\int_t^\infty (1-\gamma)\rho V_s^i \left(\log(c_s^i) - \frac{1}{1-\gamma} \log\left((1-\gamma)\, V_s^i \right) \right) ds \right].$$

In the special case $\gamma = 1$, this specification collapses to our baseline specification with logarithmic utility discussed in Section 2.[28]

3.2 Calibration

We calibrate our model such that, when we feed in a quantitatively realistic process for idiosyncratic risk, the model generates variations in output, consumption, and surpluses and aggregate risk premia that are broadly consistent with US data. For our mapping from the model to the data, one time period in the model corresponds to one year.

We take the parameters $\tilde{\sigma}^0$, ψ, σ for the exogenous $\tilde{\sigma}_t$ process from Merkel (2020),

[27] The processes a and $\mathfrak{g}$ are not individually relevant for anything of interest here, just their difference $a - \mathfrak{g}$ is.

[28] Qualitatively, the two models behave identically. However, with $\gamma = 1$, the model does not generate a sufficiently large aggregate price of risk to capture the empirically observable equity premium.

who reinterprets the idiosyncratic capital shocks as idiosyncratic TFP shocks and chooses the exogenous process parameters to match the evidence on establishment-level idiosyncratic TFP shocks reported by Bloom et al. (2018). We set the fraction $\bar{\chi}$ of idiosyncratic risk that must be retained by insiders to one half in line with the evidence on the contribution of private equity to the wealth of US investors reported by Angeletos (2007). We set the capital adjustment cost parameter ϕ to 6 (currently uncalibrated).

We choose the remaining six parameters $\gamma, \rho, \alpha_0, \alpha_1, \nu_0, \nu_1$ such that the model generates values for the volatilities of output, consumption and the surplus-output ratio, the average consumption-output, surplus-output, capital-output and debt-output ratios as well as the equity premium and equity sharpe ratio that are broadly in line with the empirical evidence.

Table 1: Parameter Choice

parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.29
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.15
σ	$\tilde{\sigma}_t^2$ volatility	0.037
$\bar{\chi}$	undiversifiable idio. risk	0.5
ϕ	capital adjustment cost	6
γ	risk aversion	7.5
ρ	time preference	0.17
α_0	C/K intercept	0.59
α_1	negative of C/K slope	0.2
ν_0	negative of $\breve{\mu}^{\mathcal{B}}$ intercept	0.085
ν_1	$\breve{\mu}^{\mathcal{B}}$ slope	0.25

Table 1 summarizes our parameter choice and Table 2 summarizes the quantitative model fit. We report data moments both for our full sample (1966–2019) and for the post-1985 period, as only during the latter US government debt has been a negative-β asset.[29]

The model generates output and consumption volatility that is slightly higher than but of similar magnitude as in the data. Government surpluses are slightly less volatile in our model, but they are also more correlated with output due to the fact that everything in our model is driven by a single shock. In total, the component of surplus

[29] This is mainly due to the stagflation episode of the 1970s. Our simple model with a single state variable cannot account for occasional stagflation episodes.

Table 2: Quantitative Model Fit

moment: symbol	moment: description	model	data: full sample (1966–2019)	data: post 1985
$\sigma(Y)$	output volatility	0.019	0.014	0.010
$\sigma(C)$	consumption volatility	0.010	0.008	0.007
$\sigma(S/Y)$	surplus volatility	0.004	0.009	0.011
$\rho(Y,C)$	correlation of output and consumption	0.977	0.826	0.83
$\rho(Y,S/Y)$	correlation of output and surpluses	0.927	0.471	0.710
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.667	0.615	0.614
$\mathbb{E}[S/Y]$	average surplus-output ratio	0.007	0.007	0.004
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.206	≈ 3	
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.672	0.578	0.714
$\mathbb{E}[dr^E - dr^{\mathcal{B}}]$	average equity premium	5.6%	$\approx$ 6.4%	
$\frac{\mathbb{E}[dr^E - dr^{\mathcal{B}}]}{\sigma(dr^E - dr^{\mathcal{B}})}$	equity sharpe ratio	0.436	≈ 0.5	

Notes: $\sigma(x)$ denotes the standard deviation of x and $\rho(x,y)$ denotes the correlation of x and y, both at a quarterly frequency. Inputs x and y are HP-filtered with smoothing parameter 1600. For $x, y \in \{Y, C\}$, we take logarithms before filtering. $\mathbb{E}[x]$ denotes expectations over the ergodic model distribution, inputs x are *not* HP-filtered. Y: (aggregate) output, C: consumption, S: primary surplus, q^K, q^B, dr^B, dr^E are defined as in Section 2.

variation that is systematically comoving with output is thus approximately as volatile as in the data.[30]

Table 2 also shows that our model does a good job at matching a number of important first moments, including the equity premium (and the equity sharpe ratio). The latter is particularly important because it verifies that our model is capable of generating realistic aggregate risk premia.

3.3 Equilibrium Dynamics of Bond and Capital Values

Figure 1 illustrates the equilibrium dynamics of the value of the government bond stock q^B (blue line) and the value of the capital stock q^K (red line) per unit of capital in the economy by plotting these valuations as a function of the state variable $\tilde{\sigma}$. The gray shaded area depicts the stationary distribution of $\tilde{\sigma}$. q^B is strictly increasing in idiosyncratic risk whereas q^K is strictly decreasing. Because output comoves negatively with $\tilde{\sigma}$ by construction, these monotonicity patters imply that bond valuations are counter-cyclical whereas capital valuations are pro-cyclical. It is this counter-cyclical valuation that makes government bonds a good safe asset. We analyze the source of the counter-cyclicality in the following subsection.

[30] It is this part of surplus variation that ultimately matters for asset pricing.

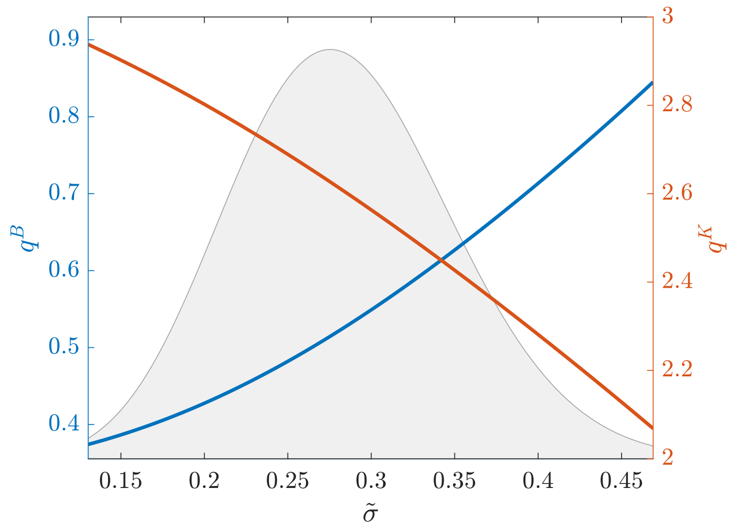

Figure 1: Equilibrium asset valuations q^B (blue line, left scale) and q^K (red line, right scale) as a function of idiosyncratic risk $\tilde{\sigma}$. The gray shaded area in the background depicts the (rescaled) ergodic density of the state variable $\tilde{\sigma}$.

3.4 Analyzing the Two Bond Asset Pricing Terms Separately

We now consider the two terms in the government debt valuation equation derived from the individual perspective (equation (10)). Figure 2 plots the two present values[31]

$$q^{B,CF}(\tilde{\sigma}) := \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) s_t K_t dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0\right] / K_0$$

$$q^{B,SF}(\tilde{\sigma}) := \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) (1-\vartheta_t)^2 \gamma \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0\right] / K_0$$

for our calibrated model. The blue solid line shows the present value of future primary surpluses (cash flows) $q^{B,CF}$ as a function of the single state variable $\tilde{\sigma}$. This value is strictly decreasing in idiosyncratic risk and has a low – and sometimes negative – value. Comparing the present value of surpluses $q^{B,CF}K$ in our model to the market value of government debt $q^B K$, which is represented by the black dashed line in Figure 2, reveals a large gap $(q^B - q^{B,CF})K$, a "debt valuation puzzle". In addition, when compared with the present value of surpluses $q^{B,CF}K$, the total value of government debt $q^B K$ has

[31] Relative to equation (10), here an additional factor γ appears because we no longer assume logarithmic preferences.

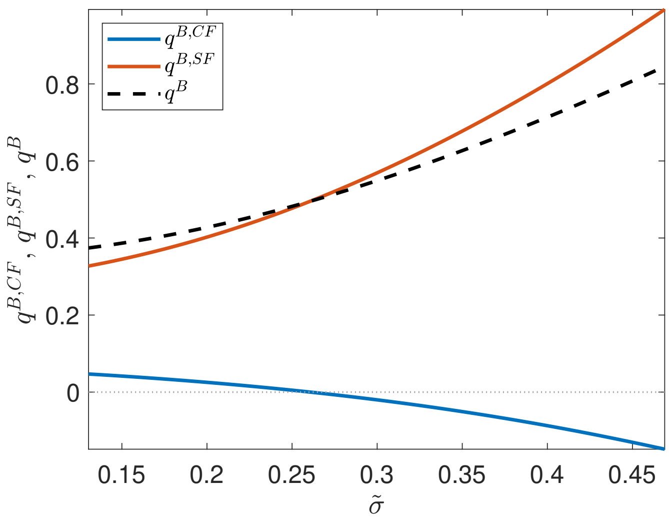

Figure 2: Decomposition of the value of government debt as a function of idiosyncratic risk $\tilde{\sigma}$. The blue solid line shows the present value of primary surpluses ($q^{B,CF}$), the red solid line the present value of service flows ($q^{B,SF}$) and the black dashed line the total value of government debt (q^{B}), all normalized by the capital stock.

also the opposite correlation with the aggregate state. Yet, there is no puzzle from the perspective of our model: government debt is a safe asset valued for its service flow from re-trading which is represented by the component $q^{B,SF}(\tilde{\sigma})$. As the red solid line in Figure 2 shows, this value is positive, large and positively correlated with $\tilde{\sigma}_t$. This additional component dominates the overall dynamics of the value of government debt and is the reason that q^{B} appreciates in bad times despite the simultaneous drop in $q^{B,CF}$. That $q^{B,SF}$ must be positively correlated with $\tilde{\sigma}$ can also be seen from the present value equation: one can show that for our policy specification residual net worth risk $(1-\vartheta_t)\bar{\chi}\tilde{\sigma}_t$ must be strictly increasing in $\tilde{\sigma}_t$, so that an increase in idiosyncratic risk increases the value of insurance service flows from re-trading.[32]

The correlation structure apparent in Figure 2 implies that, if the two claims $q^{B,CF}$ and $q^{B,SF}$ could be traded separately, the cash flow claim would be a high-β asset, while the service flow claim would be a negative-β asset. The presence of this second, negative-β component makes government debt as a whole a negative β asset. Govern-

[32]This is not an entirely rigorous argument as it ignores changes in the discount rate. The effective discount rate in the weighted-average SDF $\int \xi_t^i \eta_t^i di$ can both increase or decrease with the aggregate state x_t depending on whether the *aggregate* risk premium increases or decreases. Note however, that the level of idiosyncratic risk does not directly matter for the effective discount rate because the risk premium on idiosyncratic risk exactly offsets the lower risk-free rate due to a precautionary motive.

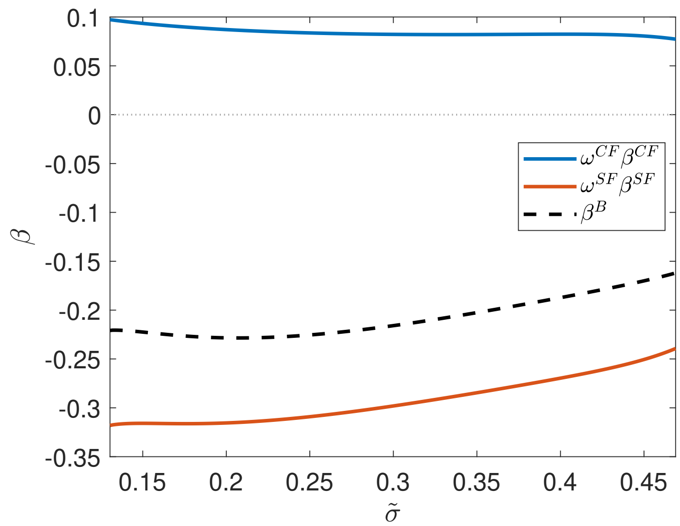

Figure 3: Conditional betas of hypothetical claims to the surplus and risk-sharing components of the government debt value

ment debt emerges as a "good friend" also with respect to aggregate shocks. Figure 3 depicts this explicitly by plotting (weighted) conditional betas for the two hypothetical assets.[33]

3.5 The Possibility of Insuring Bond Holders and Tax Payers at the Same Time

In our simple setting citizens are capital owners and bond holders. In this section, we conceptually separate each household into two sub-units, a capital owner and a government debt holder. Surprisingly, it is possible to follow a government policy that provides tax payers insurance against negative aggregate shocks and bond holders at the same time. By cutting taxes (or even granting subsidizes) for capital owners in recessions, their tax burden is positively correlated with their income providing insurance to tax payers. At the same time, the safe asset premium rises in recessions, which provides insurance to government bond holders. Importantly, this finding in our in-

[33]We define $\beta_t^j = \sigma_t^{r^j}/\varsigma_t$, where $j \in \{CF, SF\}$ and dr^j is the return on the respective component and $\sigma_t^{r^j}$ is the aggregate risk loading of that return. This definition can be interpreted as $\beta_t = -\frac{\mathrm{cov}_t(d\xi_t/\xi_t, dr_t^j)}{\mathrm{var}_t(d\xi_t/\xi_t)}$, where $d\xi_t/\xi_t$ is the SDF that discounts cash flows from $t+dt$ to time t. In addition, we weight β^j by its share $\omega^j := q^{B,j}/q^B$ of the total government debt claim.

complete market setting with a safe-asset bubble is in sharp contrast to traditional asset pricing in which either tax payers or government bond holders can be insured, as pointed out in Jiang et al. (2020).

4 Volatile, Flight-to-Safety Prone Equity Markets

The presence of idiosyncratic risk and government debt as a safe asset also has implications for equity markets. We explain in this section why the diversified equity portfolio does not emerge as a safe asset and how flight to safety can generate additional equity return volatility.

Why Stocks Are not Safe Assets. In our model, agents can hold a diversified stock portfolio. Like government bonds, this stock portfolio is free of idiosyncratic risk and thus allows agents to self-insure against idiosyncratic consumption fluctuations. However, unlike government bonds, stocks are poor aggregate risk hedges as they are ultimately claims to capital, which looses in value in recessions. This implies that stocks are positive-β assets in our model.

To understand why stock prices fall in times of high idiosyncratic risk, even though idiosyncratic equity risk can be diversified away, note that the marginal holder of capital in our model is always an insider who has to bear the increased idiosyncratic risk. As a consequence, when idiosyncratic risk goes up, so does the insider premium earned by the managing households, which is achieved by a reduction in the dividend that is paid to outside equity holders. This makes stock dividends more procyclical than production cash flows, so that stocks lose value precisely when idiosyncratic risk goes up.

When evaluating the diversified stock portfolio with regard to the two key characteristics of safe assets, the Good Friend Analogy and the Safe Asset Tautology, stocks fail to qualify as safe assets in the same way as government debt does. Stocks have the good friend characteristic only partially: stocks are valuable and liquid when an agent experiences a negative idiosyncratic shock, but due to their positive β, they are not in bad aggregate times. The positive β property and the absence of a safe asset bubble on stocks also means that stocks do not have a safe asset status in the sense of the Safe Asset Tautology, either.[34]

[34] There may be alternative equilibria which feature bubbles on stocks. We defer the discussion of those to Section 7.

Flight-to-safety Volatility. While the focus of this paper is on government bonds, our model can match the empirical mean and volatility of the excess return on the stock market in excess of government bonds. The realistic sharpe ratio is clearly a feature of recursive preferences with a high risk aversion, but the ability of our simple model to generate large return volatility in the presence of realistic levels of output variation is quite remarkable[35] and directly related to the existence of safe government bonds.

To gain intuition, let's abstract from the distinction between capital and outside equity[36] and for a moment also switch off both government spending G_t and physical capital investments I_t by putting $\mathfrak{g} = 0$ and considering the limit $\phi \to \infty$, so that $Y_t = C_t$. Then, aggregating individual households' intertemporal budget constraints yields the equation

$$q_t^K K_t + q_t^B K_t = \mathbb{E}_t \left[\int_t^\infty \frac{\int \xi_s^i \eta_s^i di}{\int \xi_t^i \eta_t^i di} Y_s ds \right]. \tag{19}$$

In standard Lucas-type models, government debt does not represent positive net wealth, $q_t^B = 0$, and thus equation (19) implies for such models that the value of the capital stock equals the present value of future output. In other words, in a Lucas-type economy, pricing the aggregate equity claim is equivalent to pricing the aggregate output claim.[37] In the presence of realistic output volatility, a large volatility in capital valuations $q_t^K K_t$ is then hard to generate (and requires substantial time variation in the SDF $\int \xi_s^i \eta_s^i di$). If we allow for $G_t, I_t \neq 0$, the puzzle tends to become even larger because consumption is smoother than output in the data.

In our model, $q^B \neq 0$ and this suggests an additional explanation for the high observed stock market volatility. When idiosyncratic risk $\tilde{\sigma}_t$ rises, there is a flight to safety that increases the value of bonds (q_t^B) and lowers the value of capital (q_t^K). Even in the

[35] This is remarkable because we work with preferences that feature a unit elasticity of intertemporal substitution (EIS). It is well-known from the long-run risk literature (e.g. CITE SOME PAPER(S)) that recursive preferences can also generate large return volatility, but *only* if the EIS is sufficiently larger than 1. In contrast, the mechanism we describe here works also for EIS ≤ 1.

[36] As we have discussed previously in this section, a state-dependent insider premium will ensure that equity values and capital values move in lockstep despite the fact that idiosyncratic equity risk can be diversified away.

[37] Because the equation results from aggregating individual intertemporal budget constraints, the SDF used in this pricing equation is again the weighted-average SDF as in the individual perspective to government bond valuation, not any market SDF (i.e. a SDF that prices all tradeable assets). Of course, in most Lucas-type models there is no idiosyncratic risk so that the two coincide.

absence of changes in the present value on the right-hand side of equation (19), this portfolio reallocation generates *flight-to-safety volatility* in capital valuations and thus in the stock market.

To get a sense how much flight-to-safety volatility matters quantitatively, we compare the excess stock return volatility in our model to the one generated by a version of the model without government debt (and primary surpluses set to zero). In that alternative version, $q_t^B = 0$ at all times and thus flight-to-safety volatility disappears.[38] To make the comparison fair, we compute excess returns in this alternative model not in excess of the risk-free rate but in excess of a (zero net supply) asset that has the same negative β as government debt in our baseline model.[39] We find that the average (annualized) excess return volatility in the alternative model would be 2.9% as opposed to 12.9% in our baseline model. We can therefore conclude that flight-to-safety volatility accounts for more than three quarters of the overall excess return volatility in our framework.

5 Contrasting Dynamic Trading Service Flows with Convenience Yields

The service flows arising from re-trading of government bonds are conceptually different from a convenience yield. A convenience yield on government debt captures the special role that government bonds play in certain transactions. It can be measured by comparing the yield on government debt with the yields on nominally safe corporate debts of equal maturity. In contrast, the service flow we emphasize in this paper does not affect the yields on government debt and safe private debt differentially. A risk-sharing service flow from re-trading can be derived from all assets that are both free of idiosyncratic risk and tradeable on liquid markets.[40] It can therefore not be measured

[38] Except for the elimination of primary surpluses ($\nu_0 = \nu_1 = 0$) and the selection of the "non-monetary" equilibrium, we keep all other parameters as in our baseline model.

[39] Specifically, we take the $\beta(\tilde{\sigma})$ function from the solution of the model with government debt and price a benchmark "bond" asset in the alternative model that has the return volatility $\sigma_t^{r^{\mathcal{B}}} = \beta(\tilde{\sigma}_t)\varsigma_t$, where ς_t is the (common) price of aggregate risk in all agents' SDF in the alternative model.

[40] In our model, while (zero net supply) privately issued bonds generate the same service flows to their holders as government bonds, the issuing agent has to "pay" those service flows by bearing more idiosyncratic risk. Only (positive net supply) government debt generates a net service flow for the economy.

by looking at yield differentials between government and safe corporate bonds.

To illustrate the difference, we augment our model so that government debt has a convenience yield. We model the source of the convenience yield by simply putting government bond holdings in agents' utility functions. Other mechanisms like collateral constraints require richer environments but would lead to the same conclusions. To keep the model extension simple, we revert back to logarithmic preferences and introduce separable logarithmic bond utility as in Di Tella (2020), that is agent i maximizes

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left((1-v)\log c_t^i + v\log b_t^i\right)dt\right],$$

where

$$b_t := (1-\theta_t^{K,i}-\theta_t^{E,i}-\theta_t^{\bar{E},i})n_t^i$$

are real bond holdings by the agent. In this augmented model, we denote the nominal interest rate paid by the government by i_t^B and continue to denote by i_t the market nominal short rate on other nominally risk-free debt that does enter the utility function. i_t is the (shadow) nominal interest rate that private agents have to pay on their nominally risk-free debt liabilities. The difference $\Delta i_t := i_t - i_t^B$ captures the convenience yield on government bonds.

We solve this augmented model in Appendix X. The model solution is almost identical to our baseline model. ι, q^B, and q^K are given by

$$\iota_t = \frac{(1-\vartheta_t)(a_t-\mathfrak{g}_t)-(1-v)\rho}{1-\vartheta_t+\phi(1-v)\rho},$$
$$q_t^B = \vartheta_t\frac{1+\phi(a_t-\mathfrak{g}_t)}{1-\vartheta_t+\phi(1-v)\rho},$$
$$q_t^K = (1-\vartheta_t)\frac{1+\phi(a_t-\mathfrak{g}_t)}{1-\vartheta_t+\phi(1-v)\rho}$$

as a function of the bond wealth share ϑ_t and the latter is determined by the dynamic equation

$$\mathbb{E}_t[d\vartheta_t] = \left(\rho+\breve{\mu}_t^{\mathcal{B}}-\Delta i_t-(1-\vartheta_t)^2\tilde{\chi}^2\tilde{\sigma}_t^2\right)\vartheta_t dt,$$

where $\Delta i_t = \frac{v\rho}{\vartheta_t}$ is the equilibrium convenience yield on government bonds. This equation differs from equation (6) only by the presence of the convenience yield term Δi_t, which raises the equilibrium level of ϑ_t.

In this augmented model, we can again price government debt according to the aggregate and the individual perspective. The resulting valuation equations are

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T\to\infty} \left(\mathbb{E}\left[\int_0^T \bar{\bar{\xi}}_t s_t K_t dt\right] + \mathbb{E}\left[\int_0^T \bar{\bar{\xi}}_t \Delta i_t \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right] + \mathbb{E}\left[\bar{\bar{\xi}}_T \frac{\mathcal{B}_T}{\mathcal{P}_T}\right] \right)$$

according to the aggregate perspective and

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) s_t K_t dt\right] + \mathbb{E}\left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di\right) \left(\Delta i_t + (1-\vartheta_t)^2 \tilde{\chi}^2 \tilde{\sigma}_t^2\right) \frac{\mathcal{B}_t}{\mathcal{P}_t} dt\right]$$

according to the individual perspective. From the latter, individual, perspective, the service flows from bonds in the utility function (captured by Δi_t) and from self insurance through re-trading (captured by $(1-\vartheta_t)^2 \tilde{\chi}^2 \tilde{\sigma}_t^2$) appear symmetrically. However, the aggregate perspective reveals an asymmetry. The convenience yield still enters the valuation explicitly as a service flow term. In contrast, the self-insurance service flows enter implicitly through a lower discount rate in $\bar{\bar{\xi}}_t$ due to precautionary savings and – potentially – through a the bubble term.

The terms arising from the aggregate perspective are the ones that are typically measured in empirical asset pricing. The best an empirical researcher can do when estimating a SDF based on aggregate asset price data is to identify $\bar{\bar{\xi}}_t$. When looking at yield differences between safe corporate and government bonds, the empirical researcher identifies an estimate of Δi_t. The importance of self-insurance service flows can only be determined indirectly, e.g. by finding a bubble component.[41] This is our interpretation of the empirical results in Jiang et al. (2019).

6 Mining the Bubble: The Debt-Laffer Curve

When government debt is a safe asset, the potential bubble on government debt represents a fiscal resource that can be "mined" for revenue as a substitute for taxation (compare Brunnermeier et al. (2020a)). Indeed, if the government was to choose

[41] The presence of a bubble component in the aggregate perspective means that even at the low discount rates implied by $\bar{\bar{\xi}}_t$, cash flows $s_t K_t$ and convenience yield service flows $\Delta i_t \mathcal{B}_t/\mathcal{P}_t$ are insufficient to explain the total value of government debt. The same always remains true if we discount at the higher rates implied by $\int \xi_t^i \eta_t^i di$, so that the self-insurance service flow must explain the gap.

a permanently positive bond growth in excess of interest payments $\breve{\mu}_t^{\mathcal{B}} > 0$ (and thus permanently negative primary surpluses), the value of government debt could still remain positive despite the negative present value of primary surpluses. This can be seen from both perspectives to debt valuation discussed in Section 2.3: in the individual perspective, equation (10), the value of debt remains positive despite negative surpluses if the service flow term is sufficiently large, in the aggregate perspective, equation (10), the same conclusion holds if a positive bubble term offsets the negative surplus term.

Our model therefore implies that the government may be able to finance government expenditures by "mining the bubble" without ever raising taxes for it. It can do so if undiversified idiosyncratic risk is sufficiently severe (high $\bar{\chi}\tilde{\sigma}$) such that even in the absence of positive surpluses government debt retains a positive value because of a bubble component.[42]

If this condition is satisfied, does the existence of a bubble imply that the government faces no budget constraint and can expand spending without limits? The answer is of course no as real resources are still finite and the real value of government debt reacts to the policy choice. Specifically, primary deficits per unit of capital are given by[43]

$$-s_t = \breve{\mu}_t^{\mathcal{B}} q_t^B.$$

The first factor, $\breve{\mu}_t^{\mathcal{B}}$, measures revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond holders is diluted by the issuance of new bonds, i.e., a higher $\breve{\mu}_t^{\mathcal{B}}$ represents a tax on existing bond holders. The second factor, q_t^B, is the tax base, the real value of existing debt (per unit of capital). If this tax base reacts negatively to an increase in $\breve{\mu}_t^{\mathcal{B}}$, a standard Laffer curve intuition emerges.

The negative reaction of the tax base is indeed the case and easiest to see in steady state when $\tilde{\sigma}$ is constant. Then q^B is explicitly given by (for $\gamma = 1$; compare Section 2.4)

$$q^B = \frac{\left(\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)\left(1 + \phi\left(a - \mathfrak{g}\right)\right)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\bar{\chi}\tilde{\sigma}}.$$

There are two reasons why higher deficits decrease q^B. First, there is a direct effect from

[42]In steady state, this can be expressed as a parameter condition: $\bar{\chi}\tilde{\sigma} > \sqrt{\rho/\gamma}$. In the full model, there is no such simple bubble existence condition anymore.

[43]This equation follows immediately from the government budget constraint.

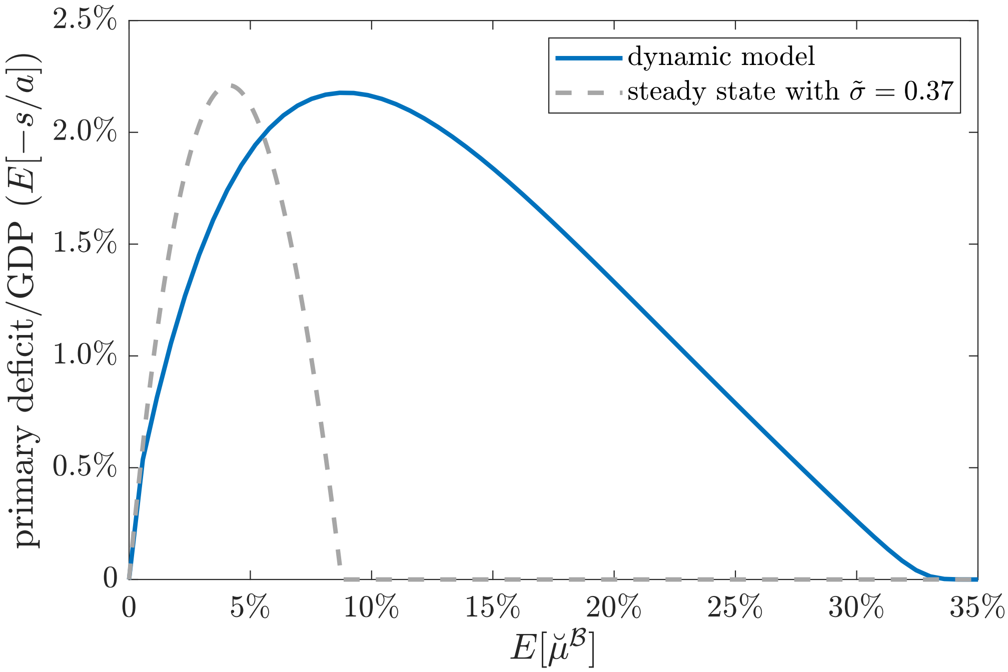

Figure 4: Debt Laffer curve for dynamic model and in steady state when there is a bubble on government debt. $\tilde{\sigma}$ for the steady state model is increased to $1.27\tilde{\sigma}^0 = 0.37$ to match the Laffer curve peak of the dynamic model.

increasing $\breve{\mu}^{\mathcal{B}}$. This emerges because higher debt growth distorts the portfolio choice between government bonds and capital, making capital more attractive and thereby lowering the fraction ϑ of wealth that originates from bond wealth. If additional deficits are used to lower the output tax rate τ, this is the only effect. However, if additional deficits are used to fund government spending by raising $\mathfrak{g}$, q^B decreases again due to the presence of the term $a - \mathfrak{g}$ (at least if $\phi > 0$). This second effect is a consequence of the resource constraint (2): when the government claims a larger share of output, consumption has to decline, which lowers all asset values symmetrically.[44]

Outside of the steady state, there is no closed-form solution for q^B anymore, but the same Laffer curve logic still applies. The blue line in Figure 4 depicts the "Debt Laffer Curve" for the calibrated dynamic model from Section 3. Specifically, this figure plots the average deficit-GDP ratio that can be sustained for different debt growth policies of the form (18) with identical ν_1 (identical cyclicality of debt growth and surpluses) but varying ν_0, i.e. the average level of (interest-adjusted) debt growth varies across

[44]This intuition breaks down for $\phi = 0$ as then agents can convert existing capital goods freely into consumption goods and instead the growth rate is reduced.

different policies on the x-axis. The assumption in Figure 4 is that $\mathfrak{g}$ remains unchanged, so that larger deficits imply smaller output taxes.

In Figure 4, if the bubble is mined too aggressively so that the average $\breve{\mu}^{\mathcal{B}}$ exceeds 8.7%, the government fails to raise additional real revenues. In particular, there is a limit to bubble mining and the government still faces a constraint on real spending. Our calibrated model suggests that the average primary deficit that can be sustained by bubble mining is bounded above by 2.2% of GDP.

It turns out that the negative β property is very important for the qualitative and quantitative shape of the Laffer curve depicted in Figure 4. If we abstracted from counter-cyclical idiosyncratic risk and considered a constant level of $\tilde{\sigma}_t = \tilde{\sigma}^0$ instead, no permanent deficit could be sustained as the steady-state bubble existence condition $\bar{\chi}\tilde{\sigma}^0 > \sqrt{\rho/\gamma}$ is not satisfied for our calibration.

To further understand the importance of the negative β property, we ask by how much we would have to increase the steady-state level of idiosyncratic risk to generate a Laffer curve with the same maximum level of deficits as in the dynamic model. The answer to this question is that idiosyncratic risk would have to be 27% larger than in the stochastic steady state of the dynamic model. The resulting steady-state Laffer curve is depicted by the gray dashed line in Figure 4. The comparison with the blue line reveals another difference between the dynamic and the steady state model: in the steady-state model the Laffer curve is steeper, so that the tax base is more quickly eroded as the government dilutes the claims of existing bond holders at a faster rate. Instead, in the dynamic model, agents hold on to some bonds even at very large levels of average (interest-adjusted) debt growth rates of more than 10% despite the high inflation rates that they imply. The reason is that the insurance against adverse aggregate events makes bonds attractive for agents even if they pay negative rates of return on average.

Reis (2020) studies a steady state model and derives the point of "bubble mining" $\breve{\mu}^{\mathcal{B}}$ when debt becomes worthless, i.e. when our Debt Laffer Curve turns negative. In his model his safe asset does not have a negative β.

7 Alternative Equilibria, Loss of Safe Asset Status, and Debt Sustainability Analysis

Government debt as a bubbly safe asset is only one equilibrium next to possible other equilibria. Does this mean that the safe asset status is a fragile arrangement? What ensures that the bubble does not burst and that we do not end up with the standard bubble-free real debt valuation, wherein government debt loses its safe-asset status? In this section we discuss how other possible equilibria would look like and argue that the government's taxation power gives government debt a natural advantage as a safe asset.[45] However, a necessary requirement for this natural advantage to materialize is that the government has the fiscal capacity and the ability to commit to taxation to defend the safe asset status whenever it is threatened. An assessment of the fiscal capacity and commitment ability to defend the safe asset status should therefore be an important ingredient in any debt sustainability analysis.

7.1 Bubble-free Equilibria and Off-equilibrium Fiscal Capacity

A bubble on government debt in our model can only exist if the terminal value term $\mathbb{E}\left[\bar{\xi}_T \frac{\mathcal{B}_T}{\mathcal{P}_T}\right]$ in the debt valuation equation from the aggregate perspective, equation (7), does not converges to zero. Because the debt-to-output ratio must be bounded along any equilibrium path,[46] this terminal value is up to a proportionality constant bounded by $\mathbb{E}\left[\bar{\xi}_T Y_T\right]$, where $Y_T := a_T K_T$ is output. The expected growth rate of $\mathbb{E}\left[\bar{\xi}_T Y_T\right]$ over a small dt-interval is $r_t^f + \varsigma_t \sigma_t^Y - g_t$, where σ_t^Y is the risk loading of output on the aggregate shock dZ_t. Consequently, a bubble can clearly not exist if on average the discount rate adjusted for the output risk premium exceeds the growth rate of the economy, $r_t^f + \varsigma_t \sigma_t^Y > g_t$. Nothing specific about the nature of government debt was used in this argument, so that it is immediately clear that under this condition also other bubbles cannot exist.

Because government policy affects discount rates, the previous considerations im-

[45] We keep the analysis largely at a verbal level in this section. Our arguments rest on the formal analysis of equilibrium multiplicity and uniqueness of Brunnermeier et al. (2020a) in the context of a steady-state version of our model.

[46] This is the case because a larger value of government debt generates a consumption demand from a wealth effect and total consumption is bounded by total available resources (equation (2)).

ply that regardless of the properties of the environment, there is always a government policy that can eliminate all bubbles: use taxes to generate primary surpluses that are a constant fraction $x > 0$ of output. Then by equation (7) and the fact that the bubble term must be nonnegative,

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} \geq x\mathbb{E}\left[\int_0^\infty \bar{\xi}_t Y_t dt\right].$$

Because the total value of debt must be finite,[47] the integral on the right must converge which implies $\mathbb{E}\left[\bar{\xi}_T Y_T\right] \to 0$ as $T \to \infty$. Economically, this is the case because as the value of debt becomes large relative to the value of capital, i.e., as ϑ approaches 1, residual idiosyncratic risk in agents' portfolios disappears which drives up discount rates beyond the threshold level at which bubbles can still exist. We conclude from these considerations that there can never be bubbles if the surplus-to-output ratio s_t/a_t is always positive and bounded away from zero. One can show that then indeed the equilibrium is unique (Brunnermeier et al., 2020a).

In this no bubble equilibrium with positive surpluses, government debt can still be a safe asset, however. While the bubble term in the aggregate perspective, equation (7), disappears, government debt still provides larger self-insurance service flows in recessions when $\tilde{\sigma}_t$ is high.[48] If these counter-cyclical service flows remain sufficiently important, they can turn government debt into a low-β or even negative-β asset despite the pro-cyclical nature of the surplus stream. In other words, such a positive surplus policy would turn government bonds into a fundamentally safe asset whose safe asset status does not require the continued belief of market participants in its safety. However, this policy would give up any revenues from bubble mining and it would also provide less insurance to tax payers in recessions.

If, in the absence of such tight fiscal policy, bubbles can exist, then there is always also a no bubble equilibrium. This is easiest to see if the government plans to never generate positive surpluses by choosing a nonnegative $\check{\mu}_t^{\mathcal{B}}$ throughout. If agents no longer believe that they can pass on the debt to someone else in the future, then it becomes worthless for them today, q^B drops to zero and the government does not collect any revenue by issuing more bonds.

[47] Otherwise there would again be an infinite consumption demand from a wealth effect such that the goods market does not clear.

[48] In this case, these service flows do not result anymore in a larger bubble from the aggregate perspective but only affect equation (7) through a lower discount rate in $\bar{\xi}_t$ due to the precautionary savings motive.

In addition to this no bubble equilibrium, there are many inflationary equilibria in between the stationary bubble equilibrium and the no bubble equilibrium. In all of these, the initial bubble is smaller than in the stationary bubble equilibrium and its value shrinks over time, so that it disappears asymptotically.

The presence of these alternative equilibria means that whenever government debt enjoys the benefit of a safe asset bubble, private agents could at any time coordinate on one of these alternative equilibria. Government debt would then (partially) lose its safe asset status. Does this mean that a bubbly safe asset status is inherently fragile or are there government policies that could avoid coordination on these other equilibria? There are such policies:

First, the government could support the current value of its debt by raising taxes so that it generates a permanently positive surplus stream that grows with total output and backs the current value. This essentially implements the no bubble policy discussed in the beginning of this section in which government debt becomes a fundamentally safe asset. However, this requires that the government has the capacity to raise taxes and it would also give up revenues from bubble mining.

Second, it is sufficient for the government to provide this tax backing *off-equilibrium*. To see this consider the case in which private investors coordinated on the belief that the bubble on government debt was smaller than in the stationary bubble equilibrium and decided to be no longer willing to hold the debt. Then the government could react by permanently reverting to a positive surplus regime in which debt is fully backed by future surpluses. Such a policy shift would generate capital gains for government bond holders and thus make the bonds so attractive ex ante that it would remain optimal for investors to hold on to their bonds.

How much *fiscal capacity* is needed to "defend" the bubble on government debt? The off-equilibrium strategy involves permanently positive primary surpluses that grow at the same rate as the economy. While the (positive) scale of these surpluses can be arbitrarily small, the fiscal authority needs the capacity and commitment to turn equilibrium deficits into surpluses before an inflationary collapse of its currency forces it to do so.[49]

[49] Ultimately, a loss of safe asset status would also force the government to give up bubble mining and reduce the deficit by inflating away the real value of government debt. However, to defend the bubble, the government must revert to surpluses and back the debt at its old, pre-inflation, value to generate capital gains for bond holders that rule out this inflationary equilibrium. It is insufficient to merely raise

7.2 Bubbles on Private Assets

So far, our discussion does not explain why the safe asset bubble is on government debt and not on any other (private) asset. Indeed, even if we restrict attention to equilibria that are not asymptotically bubble-free, equilibrium conditions still only determine the aggregate level of the bubble but not how the bubble is distributed across different assets. In theory, it is possible to have private bubbles, e.g. citizens may be able to issue pieces of paper that circulate as bubbles. Whether they are is a matter of coordination of market beliefs and thus depends on the equilibrium selection.

However, so long as agents do not face the prospects of idiosyncratic bubble creation opportunities in the future,[50] all these bubbly model equilibria lead to the same positive predictions for model aggregates with the exception that private bubbles transfer bubble mining seigniorage away from the government to private agents. In these alternative bubbly equilibria, fiscal space is therefore lower than in the equilibrium we have studied so far. If the government imposes a time 0 lump sum tax whose aggregate value equals the prevent value of private sector bubble mining revenues, uses the proceeds to purchase private assets, and holds onto its original plans for spending $\mathfrak{g}_t$, taxes τ_t, and adjusted bond growth $\breve{\mu}_t^{\mathcal{B}}$,[51] then the resulting equilibrium looks precisely like the one in which the aggregate bubble is on government bonds.[52]

How could private bubble issuance be implemented by agents in the model? Because rational bubbles cannot exist on assets with a finite maturity, the simplest way for an agent to issue a private bubble is to issue an infinitely-lived bond, e.g. a console bond. If other agents are only willing to buy such a bond at a price that does not exceed the present value of future coupon payments, then bubble creation fails and the agent has to pay back in present value exactly what he has borrowed. However, when rational bubbles are possible, then other agents could coordinate on an equilibrium in which they are willing to pay more for the bond than the present value of coupon pay-

taxes temporarily when inflation dynamics are underway to stop further inflation.

[50]Idiosyncratic bubble creation opportunities that cannot be contracted on ex ante introduce an additional source of uninsurable idiosyncratic risk and thereby affect aggregate safe asset demand.

[51]The government then has to trade in claims held against the private sector to satisfy its flow budget constraint.

[52]The wealth distribution within the private sector may be affected unless individual time 0 lump sum tax liabilities exactly equal the present value of the individual's bubble mining revenues. But these effects on the wealth distribution do not have any impact on model aggregates or the government budget in our model.

ments in the expectations that they can pass it on to other agents at a high price in the future. Such an expectation can be self-fulfilling because the self-insurance service flows derived from bond trading are proportional to the bond's real value, precisely as for government debt.

But bubbles do not need to be attached to infinite-maturity assets. Like for government debt, private agents could also mine a bubble by perpetually growing and rolling over finite-maturity debt, so that the present value of their time-T liabilities does not converge to zero as the horizon T approaches ∞. Formally, this would require that markets do not enforce a strict no Ponzi condition on individual agents as we have implicitly assumed so far. If the market does not impose a strict no Ponzi condition on agent i, agent i's transversality condition becomes $\xi_t^i n_t^i \to -n_0^{p,i} < 0$, where $n_0^{p,i}$ is the present value of bubble mining ("Ponzi wealth") that the market permits the agent in a given equilibrium. The equilibrium allocation is then equivalent to the one of a model in which the agent issues a long-lived bubble asset of value $n_0^{p,i}$ at time 0, so that $n_0^{p,i}$ is included in the agent's measured net worth n_0^i and the agent faces a strict no Ponzi condition $\liminf_{T\to\infty} \xi_T^i n_T^i \geq 0$.

Bubbles could in theory also be attached to equity claims. While in our model, outside equity claims are short-term contracts that are bubble-free, one could easily incorporate shares that circulate as bubbles by bundling the outside equity claims with any other private bubble claim like a console bond without coupon payments. Because this arrangement does not affect the asset span that agents face, it would not affect the equilibrium allocation in any way relative to a situation where the equity claim and the bubble are unbundled and can be held separately.[53] Equity bubbles would, however, affect the pricing of the aggregate stock market. If there was a bubble component on equity, the counter-cyclical valuation of the bubble would reduce the β of equity shares and turn them into safe(r) assets. As such a safe asset bubble on stocks is clearly counterfactual, these equilibria appear to be a mere theoretical curiosity.

While there is a rich set of equilibria with bubbles on private assets, ultimately government policy can eliminate such equilibria in precisely the same way as it can eliminate the no-bubble equilibrium by following an (off-equilibrium) tax policy that makes

[53] Here we maintain the assumption that there are no idiosyncratic bubble creation opportunities. If the ability to create equity bubbles was related to the agent's capacity to issue equity claims, the agent's capital holdings and thus idiosyncratic capital shocks would affect bubble creation ability and thereby alter idiosyncratic risk exposures.

its debt a more attractive safe asset than alternative private claims. For example, the government could make its off-equilibrium primary surplus stream positive and less pro-cyclical than in equilibrium. The reason why this works is the same as for the elimination of no-bubble equilibria discussed in the previous subsection. Private corporations do not have such an off-equilibrium threat to eliminate all bubbles and therefore cannot force the bubble onto their stocks.[54]

Even if a private company ever discovered a technology that generated a sufficiently safe cash flow stream growing at the same rate as the economy, the government would still have an advantage: it could use countercyclical corporate or capital income taxes to make the company's or the company's investors' after-tax cash flows more procyclical and thus the company's stock or bonds less suitable as safe assets.

8 Conclusion

In this paper we have developed a safe asset theory of government debt based on time-varying idiosyncratic insurance service flows generated by trading government bonds. Our model matches properties of US government debt qualitative and quantitatively and can resolve the empirical puzzles emphasized by Jiang et al. (2019, 2020). The theory also features a novel explanation for the large equity return volatility based on flight to safety into government bonds.

Throughout this paper we have assumed that government bonds are traded on liquid markets. The bubbly safe asset status rests on this assumption because the service flow that citizens derive from government debt is directly tied to their ability to trade it as they experience adverse shocks. The government through its central bank can engage as market maker of last resort so that citizens can trade the asset facing only small bid-ask spreads. This ensures that government debt retains the safe asset status. Private assets do not enjoy this privilege.

[54] It is a natural hypothesis that the government would want to select the equilibrium in which government debt contains the aggregate bubble and thus all seigniorage revenues are captured by the government. However, the government does not need to select this specific equilibrium. It could also select a different equilibrium with private bubbles, e.g. the government could allow certain tech firms or banks to capture some seigniorage rents.

References

Aiyagari, S. Rao and Ellen R. McGrattan, "The optimum quantity of debt," *Journal of Monetary Economics*, 1998, *42* (3), 447–469.

Angeletos, George-Marios, "Uninsured Idiosyncratic Investment Risk and Aggregate Saving," *Review of Economic Dynamics*, 2007, *10* (1), 1–30.

Bassetto, Marco and Wei Cui, "The fiscal theory of the price level in a world of low interest rates," *Journal of Economic Dynamics and Control*, 2018, *89*, 5–22.

Bewley, Truman F., "The Optimum Quantity of Money," in John H. Kareken and Neil Wallace, eds., *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, 1980, pp. 169–210.

Blanchard, Olivier, "Public debt and low interest rates," *American Economic Review*, 2019, *109* (4), 1197–1229.

Bloom, Nicholas, Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J. Terry, "Really Uncertain Business Cycles," *Econometrica*, May 2018, *86* (3), 1031–1065.

Bohn, Henning, "The sustainability of budget deficits in a stochastic economy," *Journal of Money, Credit and Banking*, 1995, *27* (1), 257–271.

Brunnermeier, Markus K. and Valentin Haddad, "Safe Assets," 2012. Working Paper.

_ and Yuliy Sannikov, "The I Theory of Money," 2016. Working Paper, Princeton University.

_ and _ , "On the Optimal Inflation Rate," *American Economic Review Papers and Proceedings*, 2016, *106* (5), 484–489.

_ , Luis Garicano, Philip Lane, Marco Pagano, Ricardo Reis, Tanos Santos, David Thesmar, Stijn Van Nieuwerburgh, and Dimitri Vayanos, "The Sovereign-Bank Diabolic Loop and ESBies," *American Economic Review Papers and Proceedings*, May 2016, *106* (5), 508–512.

_ , Sam Langfield, Marco Pagano, Ricardo Reis, Stijn Van Nieuwerburgh, and Dimitri Vayanos, "ESBies: Safety in the Tranches," *Economic Policy*, 2017, *32* (90), 175–219.

Brunnermeier, Markus, Sebastian Merkel, and Yuliy Sannikov, "The Fiscal Theory of the Price Level with a Bubble," 2020. Working Paper, Princeton University.

_ , _ , and _ , *Lecture Notes on Macro, Money and Finance: A Heterogeneous-Agent Continuous Time Approach* 2020. Princeton University.

_ , _ , and _ , "A Safe Asset Perspective for an Integrated Policy Framework," in Stephen J. Davis, Edward S. Robinson, and Bernard Yeung, eds., *The Asian Monetary Policy Forum: Insights for Central Banking*, 2021.

Caballero, Ricardo J., Emmanuel Farhi, and Pierre-Olivier Gourinchas, "The Safe Assets Shortage Conundrum," *Journal of Economic Perspectives*, August 2017, *31* (3), 29–46.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, "A Theory of teh Term Structure of Interest Rates," *Econometrica*, 1985, *53* (2), 385–408.

Dang, Tri Vi, Gary Gorton, and Bengt Holmstrom, "The Information Sensitivity of a Security," in "in" 2015.

_ , _ , Bengt Holmström, and Guillermo Ordoñez, "Banks as Secret Keepers," *American Economic Review*, April 2017, *107* (4), 1005–29.

Diamond, Peter A., "National Debt in a Neoclassical Growth Model," *American Economic Review*, 1965, *55* (5), 1126–1150.

DiTella, Sebastian and Robert Hall, "Risk Premium Shocks Can Create Inefficient Recessions," 2020. Working paper, Stanford University.

Duffie, Darrell and Larry G. Epstein, "Stochastic differential utility," *Econometrica: Journal of the Econometric Society*, 1992, pp. 353–394.

Epstein, Larry G. and Stanley E. Zin, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 1989, *57* (4), 937–969.

Gorton, Gary and George Pennachi, "Financial Intermediaries and Liquidity Creation," *The Journal of Finance*, 1990, *45* (1), 49–71.

Greenwood, Robin, Samuel Gregory Hanson, and Jeremy C. Stein, "The Federal Reserve's Balance Sheet as a Financial-Stability Tool," in "Jackson Hole Symposium," Vol. 1 Federal Reserve Bank of Kansas City Kansas City, KS 2016, pp. 335–397.

He, Zhiguo, Arvind Krishnamurthy, and Konstantin Milbradt, "A model of safe asset determination," *American Economic Review*, 2019, *109* (4), 1230–62.

Heston, Steven L., "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *The Review of Financial Studies*, 1993, *6* (2), 327–343.

Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z. Xiaolan, "The US Public Debt Valuation Puzzle," December 2019. NBER Working Paper 26583.

_ , _ , _ , and _ , "Manufacturing Risk-Free Government Debt," 2020.

Kiyotaki, Nobuhiro and John Moore, "Liquidity, Business Cycles, and Monetary Policy," April 2008. mimeo.

Krishnamurthy, Arvind and Annette Vissing-Jorgensen, "The Aggregate Demand for Treasury Debt," *Journal of Political Economy*, 2012, *120* (2), 233–267.

Li, Ziang and Sebastian Merkel, "Flight-to-Safety in a New Keynesian Model," 2020. Unpublished.

Martin, Alberto and Jaume Ventura, "The macroeconomics of rational bubbles: a user's guide," *Annual Review of Economics*, 2018, *10*, 505–539.

Merkel, Sebastian, "The Macro Implications of Narrow Banking: Financial Stability versus Growth," 2020. Working Paper, Princeton University.

Miao, Jianjun, "Introduction to economic theory of bubbles," *Journal of Mathematical Economics*, 2014, *53*, 130–136.

Reis, Ricardo, "The constraint on public debt when $r < g$ but $g < m$," 2020. Working Paper, LSE.

Samuelson, Paul A., "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy*, 1958, *66* (6), 467–482.

Tella, Sebastian Di, "Risk premia and the real effects of money," *American Economic Review*, 2020, *110* (7), 1995–2040.

Tirole, Jean, "Asset Bubbles and Overlapping Generations," *Econometrica*, 1985, *53* (5), 1071–1100.

A Appendix

A.1 Missing Steps in the Derivation of Equation (10)

We present here the missing steps in the derivation of equation (10) left out in the main text. There, we have used without proof equation (11) stated in the main text and the relationship

$$\tilde{\sigma}_t^{\Delta,i} = -\vartheta_t \bar{\chi} \tilde{\sigma}_t \tag{20}$$

for the idiosyncratic capital trading term.

Derivation of equation (11). Equation (11) is a consequence of the intertemporal budget constriant of household i,

$$n_0^i = \mathbb{E}\left[\int_0^\infty \xi_t^i c_t^i dt\right]. \tag{21}$$

To see how equation (21) implies equation (11), write initial wealth n_0^i as the sum $n_0^i = n_0^{b,i} + q_0^K k_0^i$ of initial bond wealth $n_0^{b,i} := \theta_0^i n_0^i$ and initial capital wealth $q_0^K k_0^i$. The value of initial capital wealth must equal the present value of cash flows in and out of the capital portfolio of the agent. Let X_t^i denote the total cumulative cash flow at time t, then the cash flow over a small time increment starting at time t is[55]

$$dX_t^i = \underbrace{a_t k_t^i dt}_{\text{production}} - \underbrace{\iota_t k_t^i dt}_{\text{investment}} - \underbrace{\tau_t a_t k_t^i dt}_{\text{taxes}} - \underbrace{\left(q_t^K k_t^i d\Delta_t^{k,i} + k_t^i d\langle q_t^K, \Delta_t^{k,i}\rangle\right)}_{\text{market purchases of new capital}}.$$

[55]Intuitively, the last quadratic covariation term results from the fact that market transcations are made "at the end of the period". It can be easiest seen when writing the cash flows from market purchases in pseudo discrete time notation:

$$q_{t+dt}^K k_t^i(\Delta_{t+dt}^i - \Delta_t^i) = q_t^K k_t(\Delta_{t+dt}^i - \Delta_t^i) + k_t^i(q_{t+dt}^K - q_t^K)(\Delta_{t+dt}^i - \Delta_t^i).$$

Consequently, initial capital wealth must satisfy[56]

$$
\begin{aligned}
q_0^K k_0^i &= \mathbb{E}\left[\int_0^\infty \left(\xi_t^i dX_t^i + d\langle \xi_t^i, X_t^i\rangle\right)\right] \\
&= \mathbb{E}\left[\int_0^\infty \xi_t^i k_t^i \left(a - \iota_t - \tau_t a_t\right) dt\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty \xi_t^i k_t^i q_t^K \left(\varsigma_t^i \sigma_t^{\Delta,i} + \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i} - \mu_t^{\Delta,i} - \sigma_t^{\Delta,i}\sigma_t^{q,K}\right) dt\right].
\end{aligned}
$$

Here, the last equation uses the notation $d\Delta_t^{k,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t$. Substituting the previous equation into the the intertemporal budget constraint (21) yields

$$
n_0^{b,i} = \mathbb{E}\left[\int_0^\infty \xi_t^i \left(c_t^i - k_t^i \left(a - \iota_t - \tau_t a_t\right) + q_t^K k_t^i \left(\mu_t^{\Delta,i} + \sigma_t^{\Delta,i}\sigma_t^{q,K} - \varsigma_t^i \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i}\right)\right) dt\right],
$$

which implies equation (11) if we can show that

$$
\mu_t^{\Delta,i} + \sigma_t^{\Delta,i}\sigma_t^{q,K} - \varsigma_t^i \sigma_t^{\Delta,i} - \tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i} = -\left(1 - \vartheta_t\right) \bar{\chi} \tilde{\sigma}_t \tilde{\sigma}_t^{\Delta,i}. \tag{22}
$$

We prove this equation together with equation (20) below.

Characterization of the capital trading process $\Delta^{k,i}$. In equilibrium, we have $q_t^K k_t^i = (1 - \vartheta_t) n_t^i$ and applying Ito to both sides of this equation yields

$$
\begin{aligned}
d(q_t^K k_t^i) &= q_t^K dk_t^i + k_t^i dq_t^K + d\langle q_t^K, k_t^i\rangle \\
&= q_t^K k_t^i \left(\left(\Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K} + \mu_t^{\Delta,i} + \sigma_t^{q,K}\sigma_t^{\Delta,i}\right) dt + \left(\sigma_t^{q,K} + \sigma_t^{\Delta,i}\right) dZ_t + \left(\tilde{\sigma}_t + \tilde{\sigma}_t^{\Delta,i}\right) d\tilde{Z}_t^i\right)
\end{aligned}
$$

$$
\begin{aligned}
d\left((1 - \vartheta_t) n_t^i\right) &= \left(1 - \vartheta_t\right) dn_t^i - n_t^i d\vartheta_t - d\langle \vartheta_t, n_t^i\rangle \\
&= q_t^K k_t^i \left(\left(-\rho + r_t^n - \frac{\vartheta_t \mu_t^{\vartheta}}{1 - \vartheta_t} - \frac{\vartheta_t \sigma_t^{\vartheta}}{1 - \vartheta_t}\sigma_t^n\right) dt + \left(\sigma_t^n - \frac{\vartheta_t \sigma_t^{\vartheta}}{1 - \vartheta_t}\right) dZ_t + \tilde{\sigma}_t^n d\tilde{Z}_t^i\right)
\end{aligned}
$$

[56] This is the correct present value formula for non-absolutely continuous cumulative cash flows. Again, the less familiar quadratic covariation term appearing here can easiest be understood when considering the pseudo discrete time version:

$$
PV(dX) = \sum_t \xi_{t+dt}(X_{t+dt} - X_t) = \sum_t \left(\xi_t (X_{t+dt} - X_t) + (\xi_{t+dt} - \xi_t)(X_{t+dt} - X_t)\right)
$$

Comparing terms yields

$$\tilde{\sigma}_t^{\Delta,i} = \tilde{\sigma}_t^n - \tilde{\sigma}_t = (1 - \vartheta_t)\tilde{\sigma}_t - \tilde{\sigma}_t = -\vartheta_t \tilde{\sigma}_t,$$

$$\sigma_t^{\Delta,i} = \sigma_t^n - \frac{\vartheta_t \sigma_t^\vartheta}{1 - \vartheta_t} - \sigma_t^{q,K} = \sigma_t^n - \vartheta_t \sigma^{q,B} - (1 - \vartheta_t)\sigma_t^{q,K} = 0,$$

and

$$\mu_t^{\Delta,i} = -\rho + r_t^n - \frac{\vartheta_t \mu_t^\vartheta}{1 - \vartheta_t} - \frac{\vartheta_t \sigma_t^\vartheta}{1 - \vartheta_t} \sigma_t^n - \left(\Phi\left(\iota_t\right) - \delta + \mu_t^{q,K} \right) = 0$$

Proof of equations (20) **and** (22)**.** The characterization of the idiosyncratic component of the trading process $\Delta^{k,i}$ immediately yields equation (20). To see that equation (22) holds, substitute $\mu_t^{\Delta,i} = \sigma_t^{\Delta,i} = 0$ into that equation, so that it is only left to show that

$$-\tilde{\varsigma}_t^i \tilde{\sigma}_t^{\Delta,i} = -\left(1 - \vartheta_t\right) \bar{\chi} \tilde{\sigma}_t \tilde{\sigma}_t^{\Delta,i} \Leftrightarrow \tilde{\varsigma}_t^i = \left(1 - \vartheta_t\right) \bar{\chi} \tilde{\sigma}_t.$$

The latter equation follows immediately from the fact that $-\tilde{\varsigma}_t^i$ is the idiosyncratic risk loading of $\xi_t^i = e^{-\rho t} \frac{1}{c_t^i}$, that for log utility $c_t^i = \rho n_t^i$, and $\tilde{\sigma}_t^{n,i} = (1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t$ from the net worth evolution (4) combined with the equilibrium portfolio weights.