Learning and Expectations in Dynamic Spatial Economies*

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Abstract

The impact of shocks in dynamic environments depends on how forward-looking agents anticipate the path of future fundamentals that shape their decisions. We incorporate flexible beliefs about future fundamentals into a general class of dynamic spatial models, allowing beliefs to be evolving, uncertain, and heterogeneous across groups of agents. We show how to implement our methodology to study both ex-ante and ex-post shocks to fundamentals. We apply our method to two settings: an ex-ante study of the economic impacts of climate change, and an ex-post evaluation of the China productivity shock on the U.S. economy. In both cases, we study the impact of deviations from perfect foresight on different outcomes.

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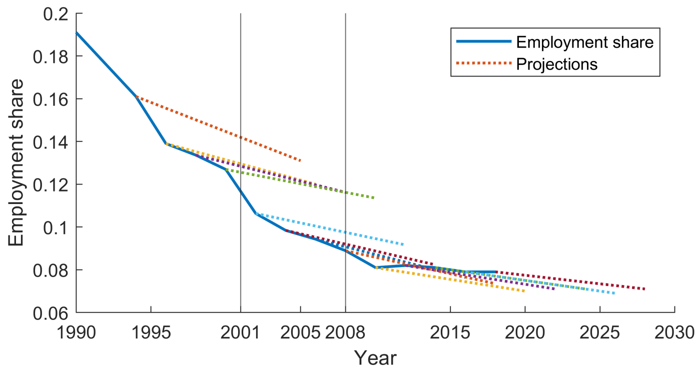
1 Introduction

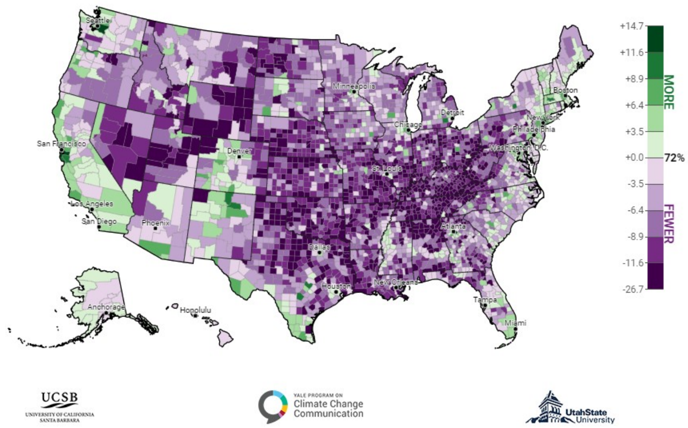
A central question in recent research is how changes in economic fundamentals shape the distribution of economic activity and impact different outcomes over time. The dynamic spatial effects of such changes are shaped by how forward-looking agents anticipate the trajectory of future fundamentals. For instance, if agents have perfect foresight, their decisions will be based on the actual path of fundamentals; if fundamentals are stochastic or if agents lack perfect foresight, then it is the agents' beliefs about future fundamentals that shape their decisions; furthermore, if the beliefs differ across groups of agents, then agents take into account not only their own beliefs but also those of other groups since the decisions of different groups interact to determine the equilibrium outcomes. In this paper, we deviate from recent spatial dynamic models that assume perfect foresight. Our main contribution is to develop a methodology with which to study the spatial effects of shocks to fundamentals in a general stochastic dynamic spatial framework, allowing the beliefs about future fundamentals to be evolving, uncertain, or heterogeneous across groups of agents. We then apply our methodology to two settings: an ex-ante study of the economic impacts of climate change, and an ex-post evaluation of the China productivity shock on the U.S. economy.

To motivate our study, Figure 1a illustrates the relevance of departing from perfect foresight in the context of the China productivity shock. As the figure shows, the manufacturing employment share in the United States steadily declined for a sustained period before flattening out around 2008. Not everyone correctly anticipated the speed and persistence of the decline, however. The dotted lines plot the 10-year projection made by the U.S. Bureau of Labor Statistics, which was too optimistic initially but was revised downward over time, converging to the actual outcomes by 2006–2008. To the extent that agents in the economy did not foresee this declining trajectory—whether because they did not anticipate productivity growth in China or failed to foresee other changes in the economy—accounting for the agents' beliefs may be important for studying the impact of the China shock. Departing from perfect foresight is also relevant for understanding the impacts of climate change. Despite growing research on the subject, it remains highly uncertain how much the temperature around the world will rise over the next decades. In addition, even the very notion of climate change is not universally accepted—according to a recent survey, almost a third of adults in the United States do not believe climate change is happening. Figure 1b shows that these climate skeptics are unevenly distributed geographically, making heterogeneous beliefs important for understanding how climate change affects spatial outcomes.

To analyze the impacts of changes in fundamentals in stochastic environments like the ones described above, we set up a dynamic spatial model with forward-looking migration decisions and a general equilibrium trade structure across locations. Without loss of generality, we assume fundamental productivities across locations follow generic stochastic processes, which can be non-stationary and history-dependent, while other fundamentals are deterministic and can be time-varying. Agents make decisions based on their shared beliefs about these processes. The stochastic equilibrium of this model is characterized by an uncountable number of equilibrium conditions. The intuition is that forward-looking agents make mobility decisions based on the



(a) U.S. Manufacturing Employment Share



(b) Share of Adults Who Think Global Warming Is Happening

Figure 1: Illustrations of Departures from Perfect Foresight Environments

Note: Panel (a) presents the manufacturing employment share in the United States during 1990–2018 (solid line) and 10-year projections by the Bureau of Labor Statistics (dotted lines). Panel (b) presents the geographical distribution of difference from the national average (72%) in the share of adults who in 2021 reported thinking global warming is happening. Source: Yale Climate Opinion (2021).

continuation value of all locations, which in turn depends on the future utility flows in all possible future productivity realizations. This process leads to an uncountable number of expected value functions one must solve for the actual migration decisions at each moment in time.

To make progress on solving the stochastic spatial equilibrium, we devise a local solution method that approximates the stochastic equilibrium around a deterministic transition path with perfect foresight. This approximation uses as inputs the deterministic transition path and the expectation of agents about future fundamentals. For any realization of the path of fundamentals, this approximation gives as output the evolution of the spatial economy in deviations from the deterministic transition path—and this output has first-order accuracy. Compared to local approximations around steady states commonly used in the study of business cycles, our approach has three advantages that are relevant to dynamic spatial economies. First, it does not rely on the existence of a steady state and therefore can be applied to non-stationary models. Second, in settings where steady states exist, it is more accurate, if the economy is in the process of transitioning to a steady state with drastically different economic landscapes from the starting point due to either the large shock under investigation or other forces not directly related to the shock (e.g., the economy in the initial period is far from the steady state). Third, the deterministic path can consider time-varying fundamentals that are not the main focus of the counterfactual analysis, which can be partialled out in our approximation—a feature especially relevant to ex-post evaluations as we discuss later on.

Building on this approach, we confront key aspects of quantifying the dynamic impact of shocks in stochastic spatial environments. In ex-post counterfactuals that investigate the impact of a *past* event, researchers observe the outcomes shaped by agents with evolving beliefs under the actual realization of the fundamentals, not the perfect foresight transition path corresponding to that realization. We propose an algorithm to recover the belief and perfect foresight paths from the observed allocations, allowing the underlying economy to have flexible beliefs as well as time-varying fundamentals such as trade and migration costs. Once the perfect foresight path has been recovered, we can approximate around it to conduct counterfactuals. Recognizing that the beliefs shaping the observed outcomes can be evolving and can deviate from perfect foresight is important. For instance, in the context of the China shock, the lack of short-term migration response to import competition can be rationalized by either high migration costs or overly pessimistic beliefs about China's productivity, which can in turn impact the subsequent counterfactual analysis.

We then generalize our methodology to incorporate uncertainty in agents' beliefs. Doing so requires solving the model with aggregate uncertainty in the presence of large state space. We extend our solution method to account for the variance-covariance of shocks and endogenous variables and show that the extended method has the second-order precision necessary to speak to the effect of uncertainty.

In addition, in environments with stochastic fundamentals, beliefs about the future might differ among groups of agents, as in the case of climate change illustrated in Figure 1b. In such settings, agents' decisions take into account not only the agents' own beliefs but also the decisions of agents with different beliefs, as these decisions, too, impact equilibrium payoffs. We extend our framework and methodology to incorporate heterogeneous beliefs. We show that our local solution method involves a curse of dimensionality akin to one in the literature of higher-order beliefs (Townsend, 1983). However, we propose alternative structures of heterogeneous beliefs that regain tractability, so an extension of the solution method described above applies.

We apply our methodology to two settings. In the first application, we study the impact of rising temperatures over 2014–2100 on welfare and the spatial allocation of economic activity in the United States. We focus on two salient features of beliefs in this environment: first, that people are generally uncertain about future temperatures, and second, that a substantial share of the population does not believe in climate change (climate skeptics). As a baseline, we solve the sequential equilibrium when climate change is perfectly anticipated. Compared to a no-climate-change scenario, perfectly anticipated climate change leads to an average welfare decrease of 1.6% in the United States. Current residents in colder states, such as those in the Northeast or Midwest, benefit; those in the South lose. Such heterogeneous impacts imply that the population gradually reallocates from the South to the North.

Incorporating uncertainty about future temperature rise increases welfare losses by 1.48 percentage points. It also leads to moderately faster spatial reallocation. The intuition is that the marginal negative impact of temperature on productivity is higher in locations where the temperature is higher. Because the South has a higher temperature today, an increase in uncertainty about future temperature leads to a larger increase in uncertainty about future productivity, incentivizing people to move away faster. Independent of the first extension, we introduce climate skeptics into the model, with their initial distribution across states given by Figure 1b. We find that the presence of climate skeptics slows down spatial reallocation, resulting in an increase in the welfare of other people (believers of climate change) in the North and a decrease in the welfare of other people in the South relative to the baseline scenario.

Our second application is an ex-post evaluation of the China productivity catch-up over 2001–2008, in which agents have evolving and potentially biased beliefs. Using a model inversion, we recover the manufacturing productivity of China and the United States and estimate a process characterizing the catch-up of China's productivity. We assume that agents make decisions based on their beliefs about the catch-up process and discipline these beliefs using the 10-year manufacturing employment projections formed over 2001–2008 and shown in Figure 1a.[1] We find that agents started out overly pessimistic about China's productivity, but these beliefs were revised and became aligned with reality by 2006. Through a counterfactual experiment, we find that the rapid productivity catch-up of China brings a 1.1% welfare gain and leads to about a 0.4 million decrease in manufacturing employment between 2001 and 2008. If we had assumed data are generated under perfect foresight, we would have found a larger decline in manufacturing employment by 2008 and slightly smaller welfare gains.

As mentioned above, our paper builds on recent spatial dynamic frameworks with perfect foresight environments, which have been used to study questions such as the spatial effects of trade shock (e.g., Caliendo et al., 2019; Rodríguez-Clare et al., 2020; Dix-Carneiro et al., 2023), transitional dynamics with migration and capital accumulation (e.g., Kleinman et al., 2023), and the impact of climate change across space (e.g., Desmet et al., 2021; Cruz and Rossi-Hansberg, 2023; Balboni, 2021; Rudik et al., 2022), among other questions. Our crucial departure is to develop a spatial framework with stochastic fundamentals and a methodology to study the spatial effects of shocks to fundamentals in environments where beliefs can be evolving, uncertain, or heterogeneous across groups of agents. Within this literature, our local approximation approach is related to Kleinman et al. (2023), who conduct first-order local approximation around steady states to extract analytical insights into the transition path in dynamic spatial models, and to Bilal (2023), who develops a perturbation method for continuous-time dynamic spatial models. Relative to these works, we develop local solutions around a transitional path with perfect foresight, which is versatile for incorporating important aspects of dynamic spatial counterfactuals in the way we described previously. In addition, our stochastic spatial framework, methodology, and applications accommodate various empirically relevant departures from perfect foresight, such as evolving,

[1] Our decision to focus on agents' beliefs about China's productivity is motivated by recent studies on the role of imports from China in U.S. manufacturing employment (Autor et al., 2013; Pierce and Schott, 2016; Handley and Limão, 2017; Caliendo et al., 2019). Our methodology applies if we assume instead that agents hold evolving beliefs about other fundamentals, such as manufacturing productivity in the United States.

uncertain, and heterogeneous beliefs. In terms of the ability to incorporate aggregate uncertainty, our paper is related to Pang and Pin (2022), who study flood risk in a spatial model with aggregate uncertainty using a deep-learning approach. Relative to this paper, our approach allows for flexible aggregate shocks—for example, each location can have its own stochastic process, which can feature long history dependence—and we are able to prove its second-order accuracy.

By allowing agents to have flexible beliefs and by providing a method for inferring these beliefs, this paper is related to studies by Dickstein and Morales (2018), Bombardini et al. (2023), and Fujiwara et al. (2020), all of whom recover agents' beliefs based on the agents' decisions. It differs in that our focus is on general equilibrium outcomes. Our model of evolving beliefs is related to a growing macro and international macro literature emphasizing the role of learning (e.g., Cogley and Sargent (2005); Kozlowski et al. (2020); Bui et al. (2022)). Aside from the focus on dynamic spatial models, our paper differs from most existing studies in that it incorporates flexible belief-updating processes about high-dimensional fundamentals in a non-stationary environment. We also provide a methodology to conduct ex-post counterfactuals based on the observed allocations.

Finally, our solution method is related to the second-order perturbation method in solving DSGE models pioneered by Schmitt-Grohé and Uribe (2004); Kim et al. (2008). The main difference is that whereas existing applications of such methods have a small number of aggregate states, our model accommodates a large number of states and sources of aggregate shocks—and flexible beliefs about these shocks, which makes it suitable for studying a wide range of questions in dynamic spatial models. Such tractability is achieved by exploiting the block-recursive property typical of a large class of dynamic spatial models and the analytical tractability within each of these blocks, as we discuss later on.

The rest of the paper is structured as follows. In Section 2, we develop a dynamic stochastic spatial framework with flexible belief structures. In Subsections 2.1 through 2.3 we describe the model. In Subsection 2.4, we construct a local approximation method to undertake counterfactual analysis with evolving beliefs, and in Subsection 2.5 we devise an algorithm for ex-post analysis. In Subsections 2.6 and 2.7, we extend the framework and methodology to account for aggregate uncertainty and heterogeneous beliefs about fundamentals. We then turn to our applications. In Section 3, we apply our framework and methodology to study uncertainty and heterogeneous beliefs about climate change, and in Section 4 we study the effects of evolving beliefs about the China shock. Section 5 concludes. All proofs and detailed derivations are relegated to the appendix.

2 A Dynamic Stochastic Spatial Model

In this section, we develop a dynamic spatial model with stochastic fundamentals.

2.1 Economic Environment

We start by describing the economic environment. The world consists of N locations, denoted by n or i. Time t is discrete and goes from 1 to T, which can be either a finite number or infinity. In each location, a continuum of individuals work and consume, and make forward-looking decisions on where to live in the next period subject to mobility frictions and idiosyncratic amenity preferences about locations. As described below, these dynamic migration decisions determine the supply of labor across locations at each moment in time. Such forward-looking labor supply decisions interact with a gravity trade structure to determine equilibrium wages and prices. To simplify the notation, we focus on a single-sector model in this section; all results generalize to a multi-sector extension with input–output linkages, which we use in our quantitative applications.

Each location n is characterized by an initial endowment of labor l_{n1} and by a set of potentially time-varying economic fundamentals $\{m_{nit}, \kappa_{nit}, z_{nt}\}$, where m_{nit} and κ_{nit} are bilateral mobility and trade frictions, respectively, and z_{nt} is the location-specific fundamental productivity. Without loss of generality, we assume that productivity z_{nt} evolves stochastically and that trade and migration costs are characterized by a deterministic path perfectly anticipated by agents. Our methodology extends to when m_{nit} or κ_{nit} is stochastic.

2.2 Stochastic Fundamentals and Beliefs

We now describe the evolution of z_{nt}. We denote the *vector* of productivity across locations in period t by $z_t \equiv (z_{1t}, z_{2t}, ..., z_{Nt})$ and the history of the productivity vector up to period t by $z^t \equiv (z_1, z_2, ..., z_t)$. We denote the set of possible outcomes for z_t in a single period by Θ and the set of possible histories for z^t by Θ^t. For example, when z_{nt} can take any positive value, Θ^t is $R_+^{N\times t}$.

Fundamental productivity z_t evolves according to a conditional probability density function (pdf) $g(z_{t+1}|z^t)$, where g specifies the functional form as well as the parameters of the stochastic process governing z_t. This pdf characterizes the likelihood of all possible histories. For example, the likelihood of history $z^{t'} \in \Theta^{t'}$ for $t' > t$ is defined by $g(z^{t'}|z^t) \equiv \prod_{i=0}^{t'-t-1} g(z_{t+i+1}|z^{t+i})$.

Agents do not necessarily know $g(z_{t+1}|z^t)$. We denote by $f(z_{t+1}|z^t)$ the conditional pdf in the eyes of period t agents after a history z^t, which we assume for now are common to all agents (we subsequently extend the model to accommodate heterogeneous beliefs). It is under the guidance of $f(z_{t+1}|z^t)$ that agents make forward-looking decisions to maximize their utility.

Through the choice of f and g, our model accommodates many setups. First, under the special case of $f = g$, agents have rational expectations. Second, both g and f can be non-stationary and time-varying, i.e., the process governing fundamentals today may differ from the process governing fundamentals in the future. Without loss of generality, this time dependence is subsumed in the dependence of f and g on z^t. In the special case in which f and g depend on z_t instead of z^t, they become Markovian processes. Third, when f differs from g, it could be either because agents are wrong about the functional form of the stochastic processes or because they gradually learn about the parameters of the function.[2] Fourth, by specifying how f varies with new information, our model captures different forms of learning. For example, agents could engage in Bayesian or myopic learning; regardless of case, they could be 'naive' learners as defined by the anticipated utility framework developed by Kreps (1998), who think that their understanding of the world is correct and will not change in the future, only to find out that they are wrong; or they could be

[2] Concretely, let the true process be $g(z_{t+1}|z^t, \phi)$, where ϕ represents the parameters governing g, and let the agents' perceived process be $f(z_{t+1}|z^t, \hat{\phi})$. Agents' beliefs might differ from the true process due to either $f \neq g$ or $\hat{\phi} \neq \phi$.

'sophisticated' in the sense that they know as new data arrive, they will revise their beliefs.[3]

2.3 Dynamic Forward-Looking Migration Decisions with Stochastic Fundamentals

The continuum of agents residing in the economy consume in their current location and make forward-looking decisions on where to live subject to mobility frictions m_{nit} and idiosyncratic taste shocks ϵ_{it}. We assume ϵ_{it} are i.i.d. realizations of a Gumbel distribution (type I Extreme Value distribution) with dispersion parameter ν (e.g., Artuç et al., 2010; Caliendo et al., 2019) and are orthogonal to the stochastic fundamental productivities.

Formally, after the realization of history z^t, the value of a location n for an agent with a realization of idiosyncratic taste shocks $\epsilon_t \equiv (\epsilon_{it}...\epsilon_{Nt})$ is given by

$$V_{nt}(z^t, \epsilon_t) = U(c_{nt}(z^t)) + \max_{\{i\}_{i=1}^N}\{\beta \mathbb{E}[V_{it+1}(z^{t+1}, \epsilon_{t+1})|z^t] - m_{nit} + \nu\epsilon_{it}\}, \tag{1}$$

where $U(c_{nt}(z^t))$ is the flow utility of the agent at location n given the history z^t, with $c_{nt}(z^t)$ being the real income defined as the ratio between wage $w_{nt}(z^t)$ and price of consumption goods $P_{nt}(z^t)$. The term $\mathbb{E}[V_{it+1}(z^{t+1}, \epsilon_{it+1})|z^t]$ is the conditional expectation over both the idiosyncratic taste shocks ϵ_{t+1} and the possible fundamental productivities at time $t+1$, and is defined by

$$\mathbb{E}[V_{it+1}(z^{t+1}, \epsilon_{it})|z^t] = \int_\Theta \left[\int_{\epsilon_{t+1}} V_{it+1}(z^{t+1}, \epsilon_{t+1}) dH(\epsilon_{t+1})\right] f(z_{t+1}|z^t) dz_{t+1},$$

where $H(\epsilon_{t+1})$ is the cumulative distribution function of $(\epsilon_{1t+1}, \epsilon_{2t+1}, ..., \epsilon_{Nt+1})$.

To reduce notation, we denote by $v_{it}(z^t)$ the expected value in period t in location i, where the expectation is taken over the idiosyncratic shocks, namely, $v_{it}(z^t) \equiv \int_{\epsilon_t} V_{it}(z^t, \epsilon_t) dH(\epsilon_t)$. Under this notation, the above equation can be written as

$$\mathbb{E}[V_{it+1}(z^{t+1}, \epsilon_{it})|z^t] = \int_\Theta v_{it+1}(z^{t+1}) f(z_{t+1}|z^t) dz_{t+1} \equiv \mathbb{E}[v_{it+1}(z^{t+1})|z^t].$$

The expectation of the value of location n ($V_{nt}(z^t, \epsilon_t)$ in (1)) over idiosyncratic draws is then

$$v_{nt}(z^t) = U(c_{nt}(z^t)) + \nu \ln\left(\sum_{i=1}^N \exp\left(\beta \mathbb{E}[v_{it+1}(z^{t+1})|z^t] - m_{nit}\right)^{1/\nu}\right), \tag{2}$$

where we use the properties of the Gumbel distribution to derive the continuation value. These properties also allow us to derive the migration share from n to i in t after a history z^t, $\mu_{nit}(z^t)$:

$$\mu_{nit}(z^t) = \frac{\exp\left(\beta \mathbb{E}[v_{it+1}(z^{t+1})|z^t] - m_{nit}\right)^{1/\nu}}{\sum_{h=1}^N \exp\left(\beta \mathbb{E}[v_{ht+1}(z^{t+1})|z^t] - m_{nht}\right)^{1/\nu}}. \tag{3}$$

The migration decisions at time t, together with the distribution of workers at the beginning

[3] Following the notation in Footnote 2, with naive agents, we have that $f_t(z^{t'}|z^t, \hat{\phi}) = \prod_{i=0}^{t'-t-1} f(z_{t+i+1}|z^{t+i}, \hat{\phi}(z^0))$, and with sophisticated agents, the pdf of agent beliefs is $f_t(z^{t'}|z^t, \hat{\phi}) = \prod_{i=0}^{t'-t-1} f(z_{t+i+1}|z^{t+i}, \hat{\phi}(z^t))$.

of period t, which is determined in period $t-1$ and denoted by $l_t^n(z^{t-1})$, govern the distribution of individuals in $t+1$, namely,

$$l_{nt+1}(z^t) = \sum_{i=1}^{N} \mu_{int}(z^t) l_{it}(z^{t-1}). \tag{4}$$

Therefore, the equilibrium conditions (2), (3), and (4) determine the evolution of the distribution of labor supply across locations in this stochastic environment.

We model the labor demand with a general equilibrium gravity trade structure with CES demand over differentiated goods (e.g., Eaton and Kortum, 2002). The assumed structure implies that the share of goods purchased by location n from location i, denoted by $\lambda_{nit}(z^t)$, is

$$\lambda_{nit}(z^t) = z_{it} \left(\frac{w_{it}(z^t)\kappa_{nit}}{P_{nt}(z^t)} \right)^{-\theta}, \tag{5}$$

where θ is the trade elasticity, κ_{nit} is the bilateral trade cost, w_{it} is the wage, and P_{nt} is the price index given by

$$P_{nt}(z^t) = \left[\sum_{i=1}^{N} z_{it} \left(w_{it}(z^t)\kappa_{nit} \right)^{-\theta} \right]^{-1/\theta}. \tag{6}$$[4]

Finally, the labor market clearing condition is given by

$$w_{nt}(z^t) l_{nt}(z^t) = \sum_{i=1}^{N} \lambda_{int}(z^t) w_{it}(z^t) l_{it}(z^{t-1}). \tag{7}$$

Given the allocation of labor, equations (5), (6), and (7) solve the static trade equilibrium. We now define the stochastic sequential equilibrium of the economy.

Definition 1. ***A stochastic sequential equilibrium of the model*** *is a set of state-contingent prices* $\{w_{nt}(z^t), P_{nt}(z^t)\}_{n=1,t=1}^{N,T}$*, allocations of goods and labor* $\{\lambda_{nit}(z^t), \mu_{nit}(z^t), l_{nt}(z^{t-1})\}_{n=1,i=1,t=1}^{N,N,T}$*, and the value of locations* $\{v_{nt}(z^t)\}_{n=1,t=1}^{N,T}$ *that satisfies the equilibrium conditions determined by the location value function (2), gross flows equation (3), law of motion of labor (4), bilateral trade shares (5), local prices (6), and labor market clearing condition (7).*

Two remarks are in order. First, unlike in settings where the aggregate state of the economy at any point in time is summarized by contemporary productivity z_t and labor allocation $\{l_{it}\}_{i=1}^N$, in this definition, the state of the economy is indexed by the history of productivity until that point (z^t). This time dependence stems from the fact that not only can $\{\kappa_{nit}, m_{nit}\}$ vary arbitrarily over time, but f can also have history dependence, which enables the model to accommodate many belief-updating processes.

Second, the presence of forward-looking decisions implies that even when solving for the decision rule for the first period, one must solve the full set of equations for all periods t and all possible trajectories z^t. To see this aspect of the stochastic equilibrium, in equation (2), the value

[4]We omit the constant in the price index in the Eaton and Kortum formulation without loss of generality.

of location n after history z^t, $v_{nt}(z^t)$, depends on $\mathbb{E}[v_{it+1}(z^{t+1})|z^t], i, = 1, .., N$, which is

$$\begin{aligned}
&\mathbb{E}[v_{it+1}(z^{t+1})|z^t] \qquad (8)\\
&= \int_{\Theta} \left[U\left(c_{it+1}(z^{t+1})\right) + \nu \ln\left(\sum_{h=1}^{N} \exp\left(\beta \mathbb{E}[v_{ht+2}(z^{t+2})|z^{t+1}] - m_{iht+1}\right)^{1/\nu} \right) \right] f(z_{t+1}|z^t) dz_{t+1}.
\end{aligned}$$

The expected continuation value $\mathbb{E}[v_{it+1}(z^{t+1})|z^t]$ thus depends on the expected future utility flows and the future expected value in all z^{t+1}, which in turn is linked to outcomes in z^{t+2} through the future expected values. This dependence continues until the end of the model, so solving the model exactly for the general setup is intractable. In what follows, we propose a local solution method for the stochastic spatial equilibrium around a deterministic transition path with perfect foresight. As we show, this solution method extends tractably to the second order, allowing it to accommodate the role of uncertainty in agent behaviors; in addition, it can be extended to incorporate heterogeneous beliefs among agents.

2.4 Local Approximation

We denote by $x_t(z^t)$ the value of outcome x at time t given a history up to t; by $\bar{x}_t$, the value of outcome x in period t in a deterministic path with perfect foresight; and by $\hat{x}_t(z^t)$, the deviations of $x_t(z^t)$ from $\bar{x}_t$. These deviations are level differences for the value of locations and log differences for all other variables. We suppress the location index in $x_t(z^t)$ to denote the vector from stacking an outcome by location. For example, let $v_{it+1}(z^t) \equiv \mathbb{E}[v_{it+1}(z^{t+1})|z^t]$ and define $v_{t+1}(z^t)$ as the vector $v_{t+1}(z^t) \equiv (v_{1t+1}(z^t), v_{2t+1}(z^t), ..., v_{Nt+1}(z^t))$. Then, under the hat notation, $\hat{v}_{t+1}(z^t) = v_{t+1}(z^t) - \bar{v}_{t+1}$.

Using these definitions, we derive $\hat{x}_t(z^t)$ as a function of $\bar{x}_t$ and $\hat{z}_t$. We start with $\hat{v}_{it+1}(z^t)$, which is the difference between equation (8) and its deterministic counterpart:

$$\begin{aligned}
\hat{v}_{it+1}(z^t) &\equiv \int_{\Theta} \left[U\left(c_{it+1}(z^{t+1})\right) + \nu \ln\left(\sum_{n=1}^{N} \exp\left(\beta v_{nt+2}(z^{t+1}) - m_{int+1}\right)^{1/\nu} \right) \right] f(z_{t+1}|z^t) dz_{t+1} \\
&\qquad - \left[U\left(\bar{c}_{it+1}\right) + \nu \ln\left(\sum_{n=1}^{N} \exp\left(\beta \bar{v}_{nt+2} - m_{int+1}\right)^{1/\nu} \right) \right] \\
&\equiv \int_{\Theta} \left[F_i\left(c_{it+1}(z^{t+1}), v_{t+2}(z^{t+1})\right) - F_i\left(\bar{c}_{it+1}, \bar{v}_{t+2}\right) \right] f(z_{t+1}|z^t) dz_{t+1} \\
&= \int_{\Theta} \left[\frac{\partial F_i}{\partial \bar{c}_{it+1}} \hat{c}_{it+1}(z^{t+1}) + \frac{\partial F_i}{\partial \bar{v}_{t+2}} \hat{v}_{t+2}(z^{t+1}) + o\left(\hat{v}_{t+2}(z^{t+1}), \hat{c}_{it+1}(z^{t+1})\right) \right] f(z_{t+1}|z^t) dz_{t+1},
\end{aligned}$$

where we have defined the terms inside the squared brackets in the first line as a function F_i. Thus, $\hat{v}_{it+1}(z^t)$ can be seen as the difference between F_i evaluated at the stochastic path versus the deterministic path, integrated over possible realizations in $t+1$. The last line of the equation is a first-order Taylor approximation of $\left[F_i\left(c_{it+1}(z^{t+1}), v_{t+2}(z^{t+1})\right) - F_i\left(\bar{c}_{it+1}, \bar{v}_{t+2}\right)\right]$ at $(\bar{c}_{it+1}, \bar{v}_{t+2})$, with an approximation error $o(\hat{v}_{t+2}(z^{t+1}), \hat{c}_{it+1}(z^{t+1}))$ that is of second order to $\hat{v}_{t+2}(z^{t+1})$ and

$\hat{c}_{it+1}(z^{t+1})$.[5] Integrating this equation over the possible path of fundamental productivities up to t from the eyes of an agent making decisions at $t = 1$, we obtain

$$\int_{\Theta^t} \hat{v}_{it+1}(z^t) f(z^t|z_1) dz^t \equiv \mathbb{E}_1 \hat{v}_{it+1} \approx \frac{\partial F_i}{\partial \bar{c}_{it+1}} \mathbb{E}_1 \hat{c}_{it+1} + \frac{\partial F_i}{\partial \bar{v}_{t+2}} \mathbb{E}_1 \hat{v}_{t+2}. \tag{9}$$

We show in Supplementary Appendix C.1 that both $\frac{\partial F_i}{\partial \bar{c}_{it+1}}$ and $\frac{\partial F_i}{\partial \bar{v}_{t+2}}$ are closed-form expressions of the outcomes in the deterministic path (e.g., real consumption, migration shares). Therefore, equation (9) delivers the expectation (in eyes of agents in $t = 1$) of the deviation from perfect foresight in period $t+1$ location values as a linear function of $\mathbb{E}_1 \hat{c}_{it+1}$ and $\mathbb{E}_1 \hat{v}_{t+2}$.

We can approximate the expected deviations in the gross mobility flows analogously. Specifically, we express the mobility flows after a history of z^t as

$$\ln \mu_{nit}(z^t) = \ln \frac{\exp\left(\beta v_{it+1}(z^t)] - m_{nit}\right)^{1/\nu}}{\sum_{h=1}^{N} \exp\left(\beta v_{ht+1}(z^t) - m_{nht}\right)^{1/\nu}} \equiv G_{ni}(v_{t+1}(z^t)).$$

We then take the first-order approximation of $G_{ni}(v_{t+1}(z^t)) - G_{ni}(\bar{v}_{t+1})$ around $\bar{v}_{t+1}$ and integrate the approximation across z^t by the likelihood $f(z^t)$ to obtain $\mathbb{E}_1 \hat{\mu}_{nit}$ as a linear function of $\mathbb{E}_1 \hat{v}_{t+1}$:

$$\mathbb{E}_1 \hat{\mu}_{nit} \approx \frac{\partial G_{ni}}{\partial \bar{v}_{t+1}} \mathbb{E}_1 \hat{v}_{t+1}.$$

Using similar procedures for the rest of the equilibrium conditions delivers a system of linear equations that takes as input $\mathbb{E}_1 \hat{z}_t$ and gives as outputs the deviations in all endogenous outcomes from their deterministic counterparts.[6] The coefficients of all deviation terms in these equations (e.g., $\frac{\partial F_i}{\partial \bar{c}_{it+1}}, \frac{\partial F_i}{\partial \bar{v}_{t+2}}, \frac{\partial G_{ni}}{\partial \bar{v}_{t+1}}$) are closed-form expressions of the outcomes in the deterministic transition path with perfect foresight. Thus, we can solve this system of equations for the expected outcomes (expectations formed according to agents' belief in the first period) on the counterfactual path, and the solution takes the form of a linear function of $\mathbb{E}_1 \hat{z}_t$. In addition to delivering the expected future outcomes, the solution to the system of equations gives us the actual outcomes in the first period. Importantly, as the approximation error in each of the linearized equations is second order in the expected deviations, and as the deviations themselves are linear functions of $\mathbb{E}_1 \hat{z}_t$, the errors in these solutions are second order in $\mathbb{E}_1 \hat{z}_t$.

Although our discussion thus far has focused on the decision (and expected outcomes) of the agents in the first period, it should be clear that such approximation can be applied to the decision of agents in any period t after any history z^t. We establish this result in the following proposition.

Proposition 1. *Given the labor allocations in period t, agents' expectation at time t about the deviations in endogenous variables at time $t' \geq t$, namely, $\left\{\mathbb{E}_t \hat{v}_{it'}, \mathbb{E}_t \hat{\mu}_{nit'}, \mathbb{E}_t \hat{l}_{it'}, \mathbb{E}_t \hat{w}_{it'}, \mathbb{E}_t \hat{\lambda}_{nit'}, \mathbb{E}_t \hat{P}_{it'}\right\}_{i=1,n=1,t'=t}^{N,N,T}$, solve a system of linear equations (in the order of $N^2 \times (T-t)$), with the input being the outcomes on the*

[5] In $\frac{\partial F_i}{\partial \bar{x}}$, we use $\partial \bar{x}$ to denote level derivatives with respect to x when x is the value of locations, and to denote log derivatives with respect to x when x is other variables. In both cases, the derivatives are evaluated at the deterministic sequential equilibrium, hence the bar in $\partial \bar{x}$.

[6] The term $\mathbb{E}_1 \hat{z}_t$ enters this system of equations through the static trade structure; see equations (5) and (6).

deterministic (perfect foresight) path and the expected deviations of the fundamental productivities from the productivities on the deterministic path $\mathbb{E}_t \hat{z}_{t'}$.

Proof. See Appendix A.1. □

Proposition 1 shows that to obtain the decision of agents in period t, instead of solving all equations in Definition 1, by approximating the nonlinear model around a deterministic transitional path starting at period t, we only need to solve a finite set of linear equations. In doing so, we also obtain the expected values for future outcomes according to agents' belief in period t, which we call the *belief path* at t in subsequent discussions. The input to the system of equations is a deterministic path with perfect foresight from t and the expected deviations of future fundamentals from the fundamentals underlying the deterministic path $\mathbb{E}(\hat{z}_{t'}|z^t), t' > t$. The perfect foresight path can be computed by building on existing methods (e.g., Caliendo et al., 2019), although later on we address key aspects related to stochastic environments. The process that governs the shocks that the researcher is interested in studying and the agent's belief about the process can be guided and informed by data, as we illustrate in our second application.

By applying Proposition 1 recursively, we can study how an economy evolves as agents change their beliefs over time along any given path of fundamentals z^T. Specifically, starting from the allocation in period t, one can solve for the perfect foresight path from t and apply Proposition 1 to obtain agents' actual migration decision in t, which moves the economy to $t+1$. Repeating this process in $t+1$ delivers the agents' decision in $t+1$, which in turn gives the allocation at the beginning of $t+2$. Iterating on this process forward until T or until the agents have perfect foresight in $T' < T$ delivers the evolution of the economy under evolving beliefs.

As an illustration, consider a model economy in which the productivity in one location is growing. Agents in the economy form beliefs about this productivity growth and make migration decisions accordingly, but their beliefs are overly optimistic. In each period, we first solve for the perfect foresight path; we then approximate around that perfect foresight for agents' actual decisions, which shifts the economy to $t+1$. Figure 2 illustrates how such an adjustment process can unfold. The vertical axis of the figure is the share of people in the location experiencing productivity growth; the dotted lines plot the shares according to agents' belief at each point in time; the solid line is the actual outcome; and the dashed lines plot the outcome if agents gain perfect foresight at different periods. As the figure illustrates, in each period, a fraction of agents move to that location, anticipating and overly optimistic about productivity growth in that location. Every period, agents are surprised by the realization of productivity and revise their beliefs. This process continues until the last period of the economy. Throughout these periods, in addition to productivity, we can allow for other fundamentals of the economy, such as trade and migration costs and regional amenities, to vary over time.

In summary, Proposition 1 allows us to obtain a first-order solution to the model with stochastic environments on any path of productivity z^T. In the subsequent sections, we show that this approach generalizes tractably to accommodate uncertainty and heterogeneous beliefs. We now explain the rationale for choosing a deterministic path as the approximation point and discuss how to construct such a path in a stochastic environment for both ex-ante and ex-post analyses.

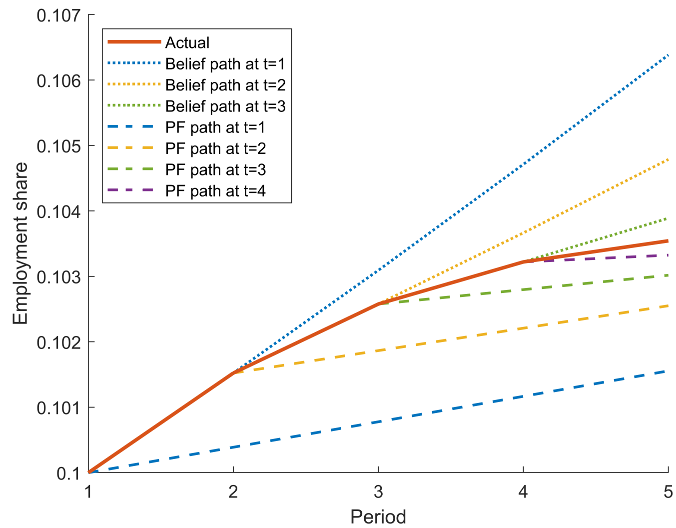


Figure 2: An Economy Under Evolving Beliefs

Note: Simulations are of an economy in which a location receives positive productivity shocks and agents in the model are too optimistic about the shocks. Plotted in the figure is the share of the population in the location receiving positive productivity shocks. The dashed lines ('PF') refer to perfect foresight paths.

Choice of approximation point. Our solution method uses a deterministic transition path with perfect foresight for approximation. In principle, any such path could be used. For example, in cases in which a model has a steady state, one could use the steady state as an approximation point. Compared to this alternative, using the deterministic transition path has three advantages.

First, it does not rely on the existence of a steady state and thus applies to non-stationary models. The model is non-stationary if, for example, it has a finite horizon, or it has an infinite horizon but the fundamentals—or agents' belief about the fundamentals—are non-stationary.

Second, it can accommodate the fact that in many applications of dynamic spatial economies, due to either the large size of the shock under investigation or other forces not directly related to the shock (e.g., the current distribution of workers being far away from the steady state), the economy might start from a point with drastically different economic landscapes than those in the steady state to which it converges. Using the steady state to approximate the decision today can lead to substantial approximation errors. One might be tempted to think that such approximation errors are of second order and do not matter materially for first-order approximations. To understand the nature of such errors, it is important to be precise about in what sense are the errors 'second order.' For concreteness, consider the approximation of a smooth single-variable function $f(x)$. Suppose we are interested in finding $f(x_2+\delta)-f(x_2)$, i.e., the impact of a change of δ on $f(x)$ around x_2. First-order approximation around x_2 gives us $f(x_2+\delta)-f(x_2) = f'(x_2)\cdot\delta + O(\delta^2)$, with the error being second order in δ, the shock of interest. If we approximate around x_1 instead, we obtain

$$
\begin{aligned}
f(x_2+\delta)-f(x_2) = f'(x_2)\cdot\delta + O(\delta^2) &= f'(x_1)\cdot\delta + [f'(x_2)-f'(x_1)]\cdot\delta + O(\delta^2) \\
&= f'(x_1)\cdot\delta + f''(x_1)\cdot(x_2-x_1)\cdot\delta + O\left((x_2-x_1)^2\cdot\delta\right) + O(\delta^2).
\end{aligned}
$$

Thus, the additional error from approximating around x_1 is $f''(x_1) \cdot (x_2 - x_1) \cdot \delta + O((x_2 - x_1)^2 \cdot \delta)$. Following this analogy, if the steady state (x_1) is far away from the actual path (x_2), then the approximation error is in the order of $(x_2 - x_1) \cdot \delta$. The error is generally first order in the shock δ unless x_2 deviates only from the steady state due to δ, in which case $x_2 - x_1$ is first order in δ and hence $(x_2 - x_1) \cdot \delta$ is second order in δ.[7]

Third, approximating around the perfect foresight path can accommodate time variations in fundamentals other than productivities such as trade and migration costs. Intuitively, our approximation can be viewed as a cross-economy comparison between a stochastic economy with a given belief structure and a deterministic one. Because all fundamentals that are not the focus of a counterfactual would be canceled in this cross-comparison, researchers do not need to recover all fundamentals that are not the focus of the counterfactual question—in a sense, this approach is a stochastic version of 'dynamic hat' algebra.[8] This feature is especially convenient in ex-post studies when time variations in other fundamentals are necessary to explain the data.

Ex-ante versus ex-post analyses. It is important to distinguish between ex-ante and ex-post counterfactuals. In ex-ante counterfactuals, the objective of the analysis is to recover how the economy evolves under a particular realization of productivity z^T, which might differ from the expected paths of z implied by either agents' beliefs or the true stochastic process. For decisions in each period, our approach requires first calculating a deterministic transition path with perfect foresight starting from that period, but it places little restriction on what perfect foresight paths to use. One approach is to simply use the perfect foresight path calculated using z^T (the dashed lines in Figure 2). An alternative is to use a perfect foresight path corresponding to agents' expected future fundamentals. In this latter case, as long as the agents' expectation is on average correct, certainty equivalence implies that up to the first order, agents' decisions in t under *that* perfect foresight path starting from t coincide with their actual decision in t. Even in this case, Proposition 1 is important as it provides the foundation for ex-post analysis, as we discuss in Section 2.5, and for incorporating uncertainty and heterogeneity beliefs, as we discuss in Sections 2.6 and 2.7.

In ex-post counterfactuals, researchers observe the evolution of the economy under a particular realization of fundamentals and aim to recover the evolution of a counterfactual economy with a different path of fundamentals. Since the actual evolution is the result of both realized and unrealized but anticipated fundamentals, it differs from the perfect foresight path corresponding to the actual realization. Thus, Proposition 1 cannot be readily used for counterfactual analysis. In the next subsection, we construct an algorithm to recover the perfect foresight path from the observed allocation and to conduct counterfactuals.

[7] This could be the case in, for example, the studies of business cycles, in which all deviations from the steady state are driven by shocks. In the application of dynamic spatial models, however, the economy can differ from the steady state due to other time-varying fundamentals or simply because the starting point of the economy is far from the steady state. In such scenarios, $x_2 - x_1$ can be much larger than δ.

[8] In contrast, to approximate the economy around a steady state, researchers have to first compute the steady state and then recover the deviations of *all* fundamentals from the steady-state values.

2.5 Recovering Perfect Foresight and Counterfactual Paths in Ex-Post Studies

Following the previous discussion, in ex-post studies, researchers do not observe the perfect foresight path corresponding to the realized fundamentals. What they instead observe is the outcome of agents' decisions under their evolving beliefs (in the example illustrated in Figure 2, the red solid line). We construct an algorithm to disentangle the role of beliefs from that of other time-varying fundamentals, recovering both the belief paths and the perfect foresight paths. We also show that when measuring the welfare of agents, the allocations in the actual and belief paths are sufficient and information on fundamentals such as migration costs is not needed. This result is useful for welfare evaluation in environments where agents do not have perfect foresight.

Suppose first that the researcher observes the actual allocation throughout all periods. Our algorithm proceeds in two steps. In the first step, we recover agents' beliefs about future paths at each point in time using a backward recursive algorithm that combines two insights. First, the observed allocation contains all information on trade costs and realized productivity, and we can use the outcomes in each period to solve for counterfactual *static trade equilibria* with different productivity beliefs or labor allocations, as in Dekle et al. (2007). Second, agents' actual migration decisions are made to maximize their utility given their beliefs about future fundamentals, including migration costs, so we can use the actual migration decision recursively to recover *belief* paths without needing to recover time-varying migration costs.

Let the actual outcomes in period t be denoted using a tilde ($\tilde{}$), i.e., $\tilde{v}_{nt}$, $\tilde{w}_{nt}$, $\tilde{P}_{nt}$, $\tilde{\lambda}_{nit}$, $\tilde{\mu}_{nit}$, $\tilde{l}_{nt}$.

(i) Starting from period T, solve for the expected outcomes of the *static trade equilibrium* in period T according to the beliefs of agents in period $T-1$, namely, $\mathbb{E}(w_T|z^{T-1})$, $\mathbb{E}(P_T|z^{T-1})$, and $\mathbb{E}(\lambda_T|z^{T-1})$, by approximating around the actual outcomes in period T. The inputs for this step are the actual outcome in T and the deviations in agents' belief in period $T-1$ about the productivity in T from the actual productivity, namely, $\mathbb{E}(\hat{z}_T|z^{T-1})$.

(ii) Append the output from the first step with agents' *actual* migration decision ($\tilde{\mu}_{niT-1}$) and the outcomes of the static trade equilibrium in period $T-1$. As $\tilde{\mu}_{niT-1}$ is decided according to agents' belief in period $T-1$, together with the output from (i), it constitutes the solution to the agents' problem in period $T-1$ that is defined in Proposition 1.[9]

(iii) Solve for the expected outcomes in periods $\{T-1, T\}$ according to the beliefs of agents in period $T-2$ by approximating around the solution to agents' problem in period $T-1$ from (ii). These outcomes include $\mathbb{E}(w_{nt}|z^{T-2})$, $\mathbb{E}(P_{nt}|z^{T-2})$, $\mathbb{E}(l_{nt}|z^{T-2})$, $\mathbb{E}(\lambda_{nit}|z^{T-2})$, $\mathbb{E}(\mu_{nit}|z^{T-2})$, and $\mathbb{E}(v_{nt}|z^{T-2})$ for $t \in \{T-1, T\}$. The inputs to this approximation are the output from (ii) and the deviations in agents' belief in period $T-2$ about the productivity in $\{T-1, T\}$ from their belief in period $T-1$, i.e., $\mathbb{E}(\ln(z_t)|z^{T-2}) - \mathbb{E}(\ln(z_t)|z^{T-1})$ for $t \in \{T-1, T\}$.

(iv) Append the output from (iii) with agents' *actual* migration decision ($\tilde{\mu}_{niT-2}$) and the outcomes of the static trade equilibrium in period $T-2$. Together, $\tilde{\mu}_{niT-2}$ and the output from

[9] The output of step (i) and $\tilde{\mu}_{niT-1}$ are both in level. However, the deviations of these level variables from the perfect foresight path starting from $T-1$ satisfy the conditions in Proposition 1. It is in this sense that they are the solution to the problem defined in Proposition 1.

(iii) constitute the solution to the agents' problem in period $T-2$.

(v) Repeat (iii) and (iv) recursively backward until the first period is reached.

This recursive algorithm recovers the solution to the agents' problem in each period and delivers expected values for all future outcomes according to agents' beliefs about productivity—in the example illustrated in Figure 2, these outcomes are represented using dotted lines.

In the second step, we recover the perfect foresight path in each period by approximating around the belief path in the same period—an application of Proposition 1. We can then approximate around the perfect foresight (or belief) path for counterfactuals, as described previously.

Importantly, in every step of the process, whenever we solve for a path as deviations from another, the deviations are always defined between variables of the two paths in the same period. Consequently, time-varying trade and migration costs are canceled out in every step, which can be seen as a stochastic version of the 'dynamic hat' algebra approach in Caliendo et al. (2019).

Extensions. In illustrating the intuition, we assumed that T is finite and that the researcher observed the actual allocation until T. We now relax these assumptions.

Suppose the researcher observes all allocations up to period $T' < T$ but not between T' and T, in which T is either a finite number or infinity. Before the above algorithm can be applied, we need to construct the path between T' and T according to the beliefs of agents in period T', taking a stand on the evolution of fundamentals over T' to T. For example, we can assume that all *other* fundamentals are either constant or change in a specific way after T', and that z still follows a stochastic process over which agents form beliefs.[10] With these assumptions, we can solve the agents' belief path based on the observed allocation in period T' using a time-difference version of the model. The intuition is that the deviations in decisions and equilibrium outcomes in each future period from the observed allocation in period T' can be written as a system of equations with the input being the deviations in fundamentals between period T' and future periods.

Armed with the belief path from T', we can start with step (iii) of the algorithm and iterate backward to recover all belief paths between 1 and T'. From these paths, we can use Proposition 1 to obtain the perfect foresight path starting from any period, or any other counterfactual paths.

In addition to the observed allocation, a key input into this exercise is agents' beliefs about fundamental productivity. In some applications, such data might not be readily available. If the researcher can gather information on agents' beliefs about endogenous outcomes that are shaped by fundamental productivity, then they can proceed in a nested approach—by choosing the belief on fundamentals that leads to the best fit between the recovered belief path and agents' actual beliefs about endogenous outcomes. We illustrate this approach in our second application.

Welfare evaluation. In applications of dynamic spatial models, researchers are interested in how a shock affects not only allocations but also agents' welfare, which can be measured using the value of locations v_{nt}. In environments with perfect foresight, such a calculation is straightforward for both ex-ante and ex-post analyses. In particular, in ex-ante analyses in which researchers have made an assumption about fundamentals, v_{nt} can be readily calculated recursively using

[10] In the case of $T = \infty$, we further assume that eventually agents have perfect foresight.

the deterministic version of equation (2). In ex-post analyses in which researchers observe the perfect foresight transition path, the real wages of locations and migration rates along the perfect foresight path can be used to evaluate welfare. Intuitively, conditional on local wages, migration choice reflects the attractiveness of other locations, accounting for migration costs. Thus, welfare can be evaluated without knowledge of underlying fundamentals.[11]

In our environment with evolving beliefs, given any path of fundamental productivity, there are at least three notions of welfare:[12] first, the expected values according to agents' beliefs at each point in time—in the example illustrated in Figure 2, the allocation associated with these values is denoted by the dotted lines; second, the values if agents had perfect foresight at each point in time, with the corresponding allocation denoted by the dashed lines in Figure 2; and last but not least, the values that correspond to the actual allocation (hereafter the actual values)—where agents make decisions based on their beliefs, which they subsequently revise as new data arrive.

The first two notions of welfare can be calculated analogously as in perfect foresight environments. For example, by evaluating the expression in footnote 11 at the expected allocations, we can calculate the first notion of welfare in $t = 1$. For the third notion of welfare, which is arguably the most relevant notion in many applications of dynamic spatial models, however, such an approach does not work. The intuition is that agents' decisions on the actual path are shaped by beliefs that are subsequently revised, so the *actual* values in t and $t+1$ are not directly linked through a forward equation in the form of (2). In the rest of this subsection, we show that the third notion of welfare can be written as the sum of the first notion of welfare, described above, and an adjustment due to the discrepancy between the actual fundamental path and the belief, and that both items can be recovered from allocations without knowledge of fundamentals.

Formally, let $\tilde{v}_{nt}$ be the *actual* value of location n in period t, and let v^t_{nt+1} be agents' expectation in period t about the value of location n in period $t+1$, namely, $\mathbb{E}_t v_{nt+1}$.[13] We have

$$\begin{aligned} \tilde{v}_{nt} &= U(\tilde{c}_{nt}) + \mathbb{E}\sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}}[\beta\tilde{v}_{it+1} - m_{nit} + \epsilon_i] \\ &= U(\tilde{c}_{nt}) + \mathbb{E}\sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}}[\beta v^t_{it+1} - m_{nit} + \epsilon_i] + \mathbb{E}\sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}}[\beta\tilde{v}_{it+1} - \beta v^t_{it+1}] \\ &= U(\tilde{c}_{nt}) - \nu\ln(\tilde{\mu}_{nnt}) + \beta v^t_{nt+1} + \sum_{i=1}^{N} \tilde{\mu}_{nit}[\beta\tilde{v}_{it+1} - \beta v^t_{it+1}]. \end{aligned} \tag{10}$$

The first equality says that the actual value of a location is the sum of the flow utility and a continuation value, with the latter depending on agents' actual choice (a function of ϵ_i, which is summarized by the indicator function) and the expectation taken over the realizations of ϵ_i. The second equality splits the continuation value into two components linearly. The third equality exploits the idea that the actual migration decision is made according to the *expected* values, which implies

[11] Formally, the value of a location under perfect foresight can be expressed as $\bar{v}_{nt} = U(\bar{c}_{nt}) - \nu\ln\bar{\mu}_{nnt} + \beta\bar{v}_{nt+1}$. Substituting this equation forward recursively leads to an expression of $\bar{v}_{nt}$ as a discounted sum of $U(\bar{c}_{nt}) - \nu\ln\bar{\mu}_{nnt}$.

[12] The path of fundamentals could be either hypothetical or (in ex-post analyses) realized.

[13] Since we are evaluating the welfare for a given path of fundamental in this exercise, we ignore z^t as an argument for all variables below.

$\mathbb{E}\sum_{i=1}^{N}\mathbb{1}_{\text{i is chosen}}[\beta v^t_{it+1} - m_{nit} + \epsilon_i] = [\beta v^t_{nt+1} - \nu\ln(\tilde{\mu}_{nnt})]$.

Equation (10) thus writes the actual values in period t as a function of the observed allocation ($\tilde{c}_{nt}$ and $\tilde{\mu}_{nnt}$), the expected values in $t+1$ according to the belief in t (v^t_{nt+1}), and the actual values in $t+1$. Note that v^t_{nt+1} can be calculated using the allocation on the belief path from period t as described previously. Therefore, starting from the last period of the model, or the period at which agents gain perfect foresight, we can iterate on equation (10) backward to obtain the actual value associated with each location in any period.

We summarize the results in this section in the following proposition:

Proposition 2. *i The actual allocations and agents' beliefs about the fundamentals in each period are sufficient to recover the expected paths according to agents' beliefs and the perfect foresight paths.*

ii The actual allocation and the belief path at each period are sufficient to recover the average welfare of agents. Information on other fundamentals is not needed.

Proof. See Appendix A.2. □

Finally, it should be clear from the discussion in this section that the general principles of our solution method for both ex-ante and ex-post analyses rely on approximating a stochastic system around a deterministic path, using the allocation on the deterministic path and the processes of the fundamentals as input. This approach can therefore accommodate many extensions, including multiple sectors, input-output linkages, and other sources of dynamics. We use an extended model with multiple sectors and intermediate goods in the quantitative applications.

2.6 Incorporating Uncertainty

In this subsection, we incorporate the uncertainty in agents' perceptions of the future into their forward-looking decisions. As it might be intuitive, doing so requires extending our methodology to accommodate a second-order approximation of the stochastic spatial equilibrium.

We denote by $\bar{x}, \bar{y}$ variables in the deterministic environment with perfect foresight, namely, $\bar{x}, \bar{y} \in \{\bar{w}_{nt}, \bar{P}_{nt}, \bar{v}_{nt}, \bar{z}_{nt}, ...\}_{n=1}^{N}$, and by $\hat{x}, \hat{y}$ the stochastic deviation terms, that is, $\hat{x}, \hat{y} \in \{\hat{w}_{nt}, \hat{P}_{nt}, \hat{v}_{nt}, \hat{z}_{nt}, ...\}_{n=1}^{N}$. To fix ideas, and following the same example as in the previous section, we derive the second-order approximation of the expected values in equation (9) as

$$\mathbb{E}_1\hat{v}_{it+1} \approx \frac{\partial F_i}{\partial \bar{c}_{it+1}}\mathbb{E}_1\hat{c}_{it+1} + \frac{\partial F_i}{\partial \bar{v}_{t+2}}\mathbb{E}_1\hat{v}_{t+2} + \frac{1}{2}\sum_{x,y}\frac{\partial F_i^2}{\partial \bar{x}\partial \bar{y}}\mathbb{E}_1\hat{x}\hat{y}. \tag{11}$$

Equation (11) shows that moving to second-order approximation requires adding covariance terms between all arguments of F_i to the first-order approximation given in equation (9). The coefficients for covariances $\mathbb{E}_1\hat{x}\hat{y}$ are the cross-derivatives of F_i with respect to x and y evaluated at the values on the deterministic path, which we denote by $\frac{\partial F_i^2}{\partial \bar{x}\partial \bar{y}}$. In Appendix A.3, we show that $\frac{\partial F_i^2}{\partial \bar{x}\partial \bar{y}}$ can be characterized as closed-form expressions of the outcomes on the deterministic path (e.g., trade shares, migration shares). We provide similar expressions for all equilibrium conditions in the stochastic economy, thereby establishing the following proposition:

Proposition 3. *Given the labor allocations in period t, agents' expectation in t about the deviations in exogenous fundamentals and endogenous variables in $t' > t$ from a deterministic path with perfect foresight* $\left\{\mathbb{E}_t\hat{z}_{it'}, \mathbb{E}_t\hat{v}_{it'}, \mathbb{E}_t\hat{\mu}_{nit'}, \mathbb{E}_t\hat{l}_{it'}, \mathbb{E}_t\hat{w}_{it'}, \mathbb{E}_t\hat{\lambda}_{nit'}, \mathbb{E}_t\hat{P}_{it'}\right\}_{i=1,n=1,t'=t}^{N,N,T}$ *and the covariance of these deviations* $\left\{e.g., \mathbb{E}_t\hat{z}_{it'}\hat{z}_{nt'}, \mathbb{E}_t\hat{v}_{it'}\hat{v}_{nt'}, \mathbb{E}_t\hat{w}_{it'}\hat{w}_{nt'}, \mathbb{E}_t\hat{w}_{it'}\hat{P}_{nt'}\right\}$ *solve a system of linear equations (in the order of* $N^2 \times (T-t)$*), with their coefficients being closed-form expressions of outcomes on the deterministic path.*

Proof. See Appendix A.3. □

Proposition 3 shows that the same system of equations as in Proposition 1, appended with second-order terms, characterizes the solution with second-order accuracy. However, whereas in the case of Proposition 1 we can simply invert the system of equations to obtain a solution, here the system is under-ranked—there are far more second-order terms in these equations than there are equations. To obtain the solution with second-order precision, we adopt a two-step procedure.

In the first step, we calculate the covariance terms $\mathbb{E}_1\hat{x}_{t'}\hat{y}_{t'}$ based on the simulations of the first-order solutions obtained using Proposition 1. Specifically, we simulate S paths of z^T according to agents' beliefs. For each simulated path $s = 1, 2, ..., S$, we use Proposition 1 recursively to obtain the outcomes of the economy, denoted by $\hat{x}_{t'}(s)$ or $\hat{y}_{t'}(s)$. We then approximate $\mathbb{E}_1\hat{x}_{t'}\hat{y}_{t'} \approx \frac{1}{S}\sum_{s=1}^{S}\hat{x}_{t'}(s)\hat{y}_{t'}(s)$. In the second step, we solve the linear conditions in Proposition 3 for the remaining terms, replacing the covariance terms with the simulated values. The resulting solution is accurate up to the second order. To give some intuition of this result, notice that

$$
\begin{aligned}
\mathbb{E}_1\hat{x}_{t'}\hat{y}_{t'} &= \frac{1}{S}\sum_{s=1}^{S}\left[\hat{x}_{t'}(s) + o\big(\hat{x}_{t'}(s)\big)\right]\cdot\left[\hat{y}_{t'}(s) + o\big(\hat{y}_{t'}(s)\big)\right] + O\left(1/\sqrt{S}\right) \\
&= \frac{1}{S}\sum_{s=1}^{S}\Big[\hat{x}_{t'}(s)\hat{y}_{t'}(s) + \underbrace{o\big(\hat{x}_{t'}(s)\big)\cdot o\big(\hat{y}_{t'}(s)\big) + \hat{y}_{t'}(s)\cdot o\big(\hat{x}_{t'}(s)\big) + \hat{x}_{t'}(s)\cdot o\big(\hat{y}_{t'}(s)\big)}_{\text{third-order terms}}\Big] + O\left(1/\sqrt{S}\right)
\end{aligned}
$$

where $o\big(\hat{x}_{t'}(s)\big)$ and $o\big(\hat{y}_{t'}(s)\big)$ represent the second-order errors in solving for $\hat{x}_{t'}(s)$ and $\hat{y}_{t'}(s)$ using Proposition 1, and $O(\frac{1}{\sqrt{S}})$ denotes the simulation error for covariance. For a large enough S, the simulation error is arbitrarily small. The only remaining sources of error are $o\big(\hat{x}_{t'}(s)\big)$ and $o\big(\hat{y}_{t'}(s)\big)$. Both errors are of second order, and they enter the approximation in multiplicative terms with other first-order or second-order terms. As a result, these errors are of the third order, and by solving the system of linear equations for the remaining variables, we arrive at a solution with second-order precision. We can then use Proposition 3 recursively to obtain the outcome of the economy under any given path of fundamentals with second-order accuracy.

Our second-order approximation relates to the perturbation method used in the study of business cycles in macroeconomics (see Kim et al., 2008; Schmitt-Grohé and Uribe, 2004 and the references thereto) with two main differences. First, while the standard perturbation method focuses on a state-space representation of the economy, our method focuses on the sequence space. As discussed previously, this difference enables us to easily accommodate non-stationary models, allowing fundamentals to vary arbitrarily and agents' beliefs about these fundamentals to depend on history in a flexible way.[14] Second, unlike in conventional applications of perturbation methods, which require researchers to calculate numerically the (first- and second-order) derivatives of policy functions with respect to the state, we take advantage of the fact that in a widely used class of dynamic spatial models, the derivatives necessary for second-order approximation can be derived as closed-form expressions of the outcomes on the approximation path. As a result, our approximation method can accommodate many locations and location-specific stochastic fundamental processes, which is relevant in a large class of spatial models.[15]

2.7 Incorporating Heterogeneous Beliefs

We have so far focused on the case in which all agents share a common belief f. In environments with stochastic fundamentals, agents might disagree about the path of future fundamentals. In such environments, agents' decision must take into account not only their own beliefs but also the beliefs of other agents as their decision, too, affects equilibrium outcomes. In this subsection, we extend our framework and methodology to a setting with heterogeneous beliefs.

Without loss of generality, assume that there are two different beliefs about the future path of fundamental productivities, each held by one group of agents.[16] We denote the two groups of agents by A and B, and their beliefs by $f^g(z_{t+1}|z^t)$, $g \in \{A, B\}$. We assume that the existence of these heterogeneous beliefs is common knowledge. In other words, agents in one group are aware of the existence of agents in the other group with a different belief but think that their own belief is the correct one. For simplicity, we assume that agents' belief types do not change, but it is feasible to allow belief types to evolve stochastically.[17] As all decisions of all agents interact in equilibrium, each group makes decisions considering the decisions of the other group guided by their own beliefs. For instance, group A agents might believe that the future productivity will be higher in a given location and move to that location. Their decision depends on the belief of group B. If B thinks differently and has a lower propensity to move to that location, then group A might have a stronger incentive to move to the location to take advantage of the lack of competition in the labor market there.

We now define the stochastic sequential equilibrium in this environment.

Definition 2. ***A stochastic sequential equilibrium with heterogeneous beliefs*** *is a set of state-contingent allocations* $\left\{\mu^g_{nit}(z^t), \lambda_{nit}(z^t), l^g_{nt}(z^{t-1})\right\}^{N,N,T}_{n=1,i=1,t=1,g\epsilon\{A,B\}}$*, prices* $\left\{w_{nt}(z^t), P_{nt}(z^t)\right\}^{N,T}_{n=1,t=1}$*, and location*

[14] To see why it might be challenging for the perturbation method to accommodate these features, note that in the most general form of f, the state space of a one-sector version of our model is $R^{t\times N}$ for period t. Our approach requires approximating $\mathbb{E}_t \hat{x}_{t'} \hat{y}_{t'}$, for all $t' > t$. The space for possible outcomes for $\hat{x}_{t'} \hat{y}_{t'}$ can be large, but by using Monte Carlo integration, the convergence rate of our algorithm is $O(\frac{1}{\sqrt{S}})$ regardless of the dimension of the space of $\hat{x}_{t'} \hat{y}_{t'}$.

[15] In the perturbation method, second-order terms can accumulate over time to a higher order, leading to an explosive path. The literature (Kim et al., 2008) has proposed 'pruning' higher-order terms to solve this problem. By calculating the covariance terms using first-order simulations, our approach avoids the accumulation of higher-order terms.

[16] Our approach generalizes to a finite number of belief types, although the computational burden increases with the types of beliefs. Even with just two belief types, however, through the composition of agents' belief types, our model can accommodate variations across locations in beliefs about fundamentals.

[17] For example, agents' belief type can follow a Markov-switching model. By allowing the switching probability to differ across locations based on local conditions such as the past fundamentals or the current share of each belief type, the model can accommodate endogenous location-specific beliefs.

values $\{v^g_{nt}(z^t)\}^{N,T}_{n=1,t=1,g\epsilon\{A,B\}}$*, such that* $w_t(z^t)$*,* $P_t(z^t)$*,* $\lambda_{nit}(z^t)$ *solve the static trade equilibrium defined by bilateral shares (5), local prices (6), and the labor market clearing condition (7), and for each group* g*,* $v^g_{nt}(z^t)$*,* $\mu^g_{nit}(z^t)$*,* $l^g_t(z^{t-1})$ *solve the dynamic migration decisions given by value function (2), gross flows equation (3), and law of motion of labor (4).*

Under Definition 2, agents correctly anticipate the decisions of all other agents in all realizations of z^t but disagree on the likelihood of z^t. In this sense, this definition corresponds to an equilibrium of full rationality. Solving this equilibrium fully is infeasible for the same reason that solving the equilibrium with homogeneous beliefs is infeasible. In fact, with heterogeneous beliefs, even the local approximation solution method established in Proposition 1 involves a curse of dimensionality akin to one in the literature of higher-order beliefs (Townsend, 1983). In what follows, we first discuss the nature of this problem and then make progress toward tractability by proposing an approximate solution that maintains the generality of the rest of our framework.

Suppose we follow the approach in Proposition 1. Consider the problem of group A, which takes as given the migration decision of group B, $\mu^B_{nit}(z^t)$. As discussed before, forward-looking decisions by group A depend on the expected future path of real wages that are in part shaped by $\mu^B_{nit}(z^t)$. Hence, solving for decisions by group A in the first period requires considering *the belief of* group A about $\mu^B_{nit}(z^t)$ for all $t \geq 2$. In turn, future decisions by group B depend on the group's belief about expected decisions by group A, introducing higher-order beliefs into the problem.

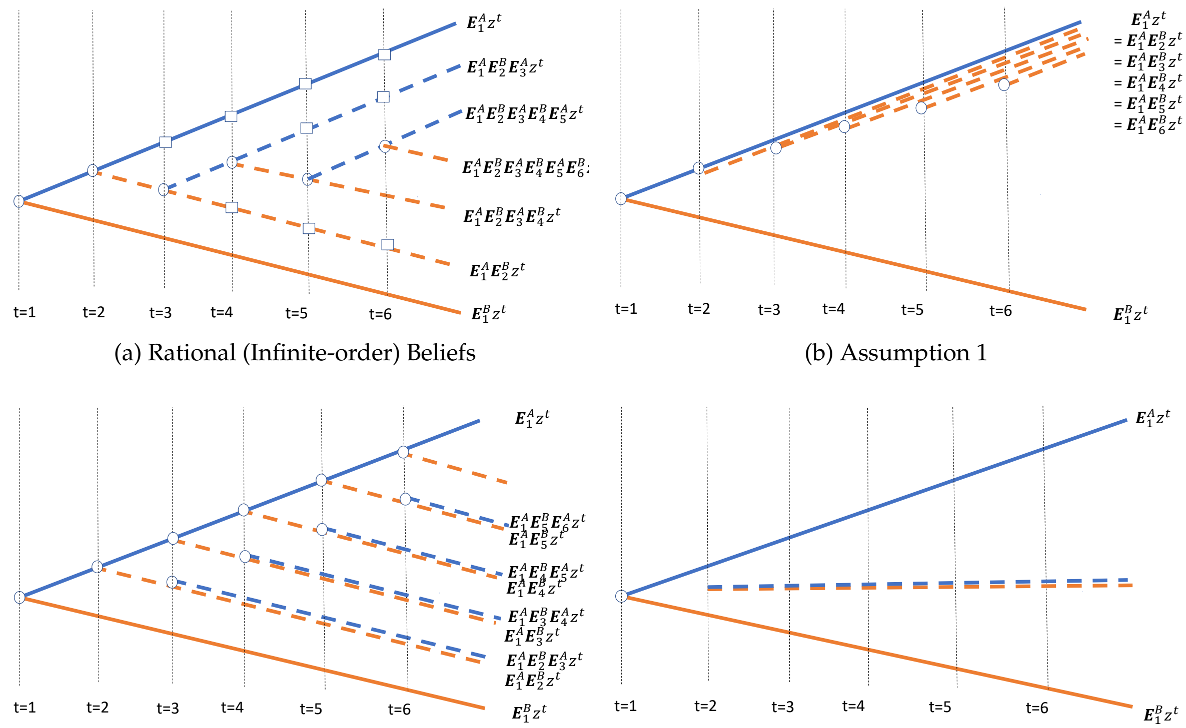
Figure 3a illustrates this dependence more concretely. In the example, in deciding what to do in the first period, group A forms expectations about the decision of group B in the second period and beyond, denoted by $\mathbb{E}^A_1(\mu^B_{int})$, $t = 2, 3, ..., T$. In turn, $\mathbb{E}^A_1(\mu^B_{int})$, $t = 2, 3, ..., T$ depends on A's belief about B's belief at each t about the future (t') decision of A, namely, $\mathbb{E}^A_1\mathbb{E}^B_t(\mu^A_{int'})$, $t' = t, 3, ..., T$. Following this rationale, $\mathbb{E}^A_1\mathbb{E}^B_t(\mu^A_{int'})$, $t' = t, ..., T$ depends on A's belief about B's belief about A's belief about B's decision in each future period $t'' > t'$, namely, $\mathbb{E}^A_1\mathbb{E}^B_t\mathbb{E}^A_{t'}(\mu^B_{int''})$, $t'' = t', ..., T$. This process continues and implies that first-order approximation to A's problem in the first period requires us to solve higher-order beliefs about future decisions, which depend on higher-order beliefs about fundamental productivity (e.g., $\mathbb{E}^A_1\mathbb{E}^B_2\mathbb{E}^A_3 z_4 ...$). In the example illustrated in Figure 3a, one must solve for the endogenous outcome on each of the nodes, which is infeasible when z_t takes continuous support.

To retain tractability, we impose three alternative structures on heterogeneous beliefs. We emphasize that these assumptions do not restrict agents' beliefs about fundamentals, but their beliefs about other people's future beliefs.

Assumption 1. *At period* t*, each type thinks (wishfully) that in* $t + 1$ *and beyond, the new data will vindicate their own belief and convince the other type.*

Assumption 2. *Agents maintain their beliefs and think the agents in the other group will also maintain their beliefs; however, group A thinks group B thinks that A will be convinced by the data in the future, and vice versa.*

Assumption 3. *Agents think their beliefs will converge in the future.*



(a) Rational (Infinite-order) Beliefs

(b) Assumption 1

(c) Assumption 2

(d) Assumption 3

Figure 3: Structures of Heterogeneous Beliefs

Panels (b), (c), and (d) of Figure 3 illustrate the structures of higher-order beliefs under these three assumptions. Under the first assumption, since each group expects that realized fundamental productivities will convince the other group that their own beliefs are correct, only up to the second-order beliefs appear in the equilibrium definition. Under the second assumption, group A thinks that group B thinks group A will be convinced and vice versa, so only up to third-order beliefs appear. Moving from Assumption 1 to Assumption 2, the complexity of the model increases. If we add more and more higher-order beliefs, the model will approach the rational stochastic equilibrium at the expense of computational complexity. Instead of going into even higher order, one can consider a setting (Assumption 3) in which both groups believe that they will converge to a common belief in the next period. These three assumptions capture a wide range of empirically relevant settings.[18] By imposing any of these assumptions, we are able to retain tractability in the stochastic spatial framework with heterogeneous beliefs, as established in Proposition 4.

Proposition 4. *Under each of the three assumptions, the local solution to the stochastic spatial model can be characterized by a system of linear equations and can be solved analogously to Proposition 1.*

Proof. See Appendix A.4. □

Intuitively, under these structures, higher-order beliefs about endogenous outcomes, which are relevant for solving the decision today, can be reduced to lower-order beliefs by the law of iterative

[18]For example, in lab experiments of coordination games, most subjects engage in less than three levels of reasoning (Mauersberger and Nagel, 2018).

expectations. For example, A's belief about B's belief about A's future decisions are simplified to A's belief about A's own future decisions under Assumption 1, and to $A's$ belief about B's future decisions under Assumption 2, thereby simplifying the system of equations that characterizes A's problem. The system of equations that characterizes B's problem can be simplified analogously. In Appendix A.4, we develop iterative algorithms to solve the problems of A and B jointly at each period, which moves the economy to the subsequent period.

Summary and extensions. In this section, we have developed a generic stochastic dynamic spatial model with flexible beliefs. We derive an approximate solution for this model around a deterministic path with perfect foresight. We then develop a methodology to study the effects of evolving beliefs on actual allocations, and to conduct counterfactuals in ex-post studies, which require researchers to reconcile observed allocations through evolving beliefs and/or other time-varying fundamentals. We also show how to introduce uncertainty and heterogeneity in agents' beliefs in dynamic spatial counterfactuals. In the rest of this paper, we apply our methodology to shed light on the role of these departures from perfect foresight through the lens of two applications. To isolate the importance of each of the departures, we focus on one at a time, but these departures can be combined, as we establish in Proposition 5.

Proposition 5. *Under Assumptions 1–3, the stochastic spatial framework with heterogeneous beliefs can be solved with second-order accuracy as described in Section 2.6, so uncertainty can be incorporated. The results described in Sections 2.6 and 2.7 can also be applied to ex-post studies in which researchers observe outcomes shaped by the decisions of agents whose beliefs are uncertain or heterogeneous, and are interested in finding the allocation corresponding to a counterfactual fundamental path.*

Proof. See Appendix A.5. □

In Appendix A.5, we discuss how, in ex-post studies with heterogeneous agents, our approach takes into account whether the migration flows by belief type are observed. There, we consider two polar cases: one in which the researcher can observe migration flows by belief type in all periods, and one in which the researcher cannot observe migration flows by belief type but can observe total bilateral migration for each period and one cross-sectional distribution of people by their belief type. We show how to conduct counterfactuals in both cases and discuss how to proceed when the data available to the researcher fall between the two cases.

We now turn to our quantitative analysis in which we explore the roles of evolving, uncertain, and heterogeneous beliefs in dynamic spatial models, applying our methodologies to two settings: climate change and China's productivity shock.

3 Ex-ante Application: Climate Change

Our first application studies the effects of climate change on welfare and the spatial distribution of economic activity in the United States. As the full impact of climate change is yet to be seen, we use this example to illustrate an ex-ante application, focusing on two departures from perfect foresight especially relevant in this context: uncertainty, in which agents are uncertain about the extent of future temperature rise, and heterogeneous beliefs, in which agents hold two views of climate change—they either believe it is happening or are skeptical and think it will not occur.

3.1 Setup and Data

We extend the baseline spatial model presented in the previous section to incorporate multiple sectors (agriculture, manufacturing, and non-tradable service), and intermediate goods, as described in Appendix B.1. We denote the sectors by $j, k \in \{A, M, S\}$. For example, v_t^{nj} is the value associated with sector j in location n in period t and z_t^{nj} is the productivity.

With a focus on the United States, we map locations in the model to the U.S. states, 27 other major countries/regions, and the synthetic rest of the world (see Appendix B). Migration can take place between U.S. states and sectors but not internationally. In addition, as one of the focuses of our quantitative analysis is uncertainty, agents' risk attitude is important. Accordingly, we specify the flow utility from consumption in location n and sector j as

$$U(c_t^{nj}) = \frac{(c_t^{nj})^{1-\sigma}}{1-\sigma},$$

where we set the risk-aversion parameter $\sigma = 3$. We treat each period as two years and choose β so the annual discount rate is 4%.

We study the role of uncertainty and heterogeneous beliefs in shaping the impact of the rising temperature over the period 2014–2100. As discussed previously, doing so requires obtaining the initial allocation (e.g., trade, migration, production) for the year 2014 and computing a deterministic path with perfect foresight starting from that allocation. We use a path of fundamentals that reflects the average effect of climate change on productivity and compute the corresponding deterministic path. We then apply Proposition 3 to study the effect of uncertainty about the path of climate change and Proposition 4 to quantify the role of heterogeneous beliefs about climate change.

We start by summarizing the data used to obtain the initial allocation for the year 2014. Additional details are presented in Appendix B.

Trade and production data. We set a trade elasticity $\theta = 4.55$ following Caliendo and Parro (2015). We obtain bilateral trade flows, gross output, value added, and input and final consumption shares across sectors and countries from the World Input-Output Database (WIOD). We obtain value added across individual U.S. states and sectors from the U.S. Bureau of Economic Analysis. We construct gross output across states and sectors using the sectoral U.S. value-added shares in gross output from the WIOD. We also construct the share of intermediate goods in production and each sector's share in final consumption for each country from the WIOD.

We obtain bilateral trade data between U.S. states and foreign countries in 2014 from the U.S. Census Bureau. We scale these data so that they match the aggregate exports and imports between the United States and other countries in the WIOD.

Bilateral trade flows across U.S. states are obtained from the Commodity Flow Survey (CFS), which contains shipment information between and within states for 12 manufacturing industries.

The closest survey year to our sample period is 2012. For the manufacturing sector, we scale the shipment values in the 2012 survey so that the aggregate shipment values (local and interstate flows combined) are consistent with the total U.S. manufacturing output that was absorbed domestically in 2014 according to the WIOD. For the agricultural sector, we impute the interstate trade flows from i to n, $X_t^{in,A}$, using the following gravity specification:

$$\ln(X_t^{in,A}) = \beta_1 \ln(X_t^{n,A}) + \beta_2 \ln(Y_t^{i,A}) + \beta_3 \ln(\text{dist}_{in}) + \beta_4 \mathbb{1}(i = n), \tag{12}$$

where $X_t^{n,A}$ is the total absorption in the agricultural sector in location n, $Y_t^{i,A}$ is the gross production in agriculture in location i, dist_{in} is the distance between the two locations, and $\mathbb{1}(i = n)$ is an indicator for $i = n$. We construct $X_t^{n,A}$ and $Y_t^{i,A}$ using the regional value-added data and the share of value added in production described above. Estimating this gravity specification using CFS data on manufacturing shipments, we obtain the coefficient $\beta_s, s = 1, 2, 3, 4$. We then use the estimated equation (12) to impute the interstate trade flows in the agriculture sector.

Migration. We set a migration elasticity of $1/\nu = 0.187$ following Caliendo et al. (2019). We construct gross mobility flows across U.S. states and sectors for 2014 using the job-to-job flows provided by the U.S. Census based on the Longitudinal Employer-Household Dynamics Database, which counts the number of movements between states and sectors. We calculate the number of stayers in each state-sector cell as the difference between total employment and outflows.

Future temperature rise and productivity damage. To construct a path of fundamentals that reflects the impact of climate change, we use temperature projection data and existing estimates of how temperature affects productivity.

Our temperature projection data are from the Climate Impact Lab, which provides temperature forecasts for world regions and the U.S. states under different carbon emissions scenarios. Each scenario contains three cases for temperature rise (the 5th, 50th, and 95th percentiles of future temperature) and in each case, the projections for three broad periods are reported: 2020–2039, 2040–2059, and 2080–2099. We use the RCP 4.5 carbon emissions scenario, which reflects an intermediate level of emission, and linearly interpolate the median temperature rise within these time frames to obtain a temperature path.

We specify the following function by which temperature affects regional productivity:

$$\begin{aligned} \ln(z_t^{nj}) &= \delta^{nj} + \alpha_1^j \bar{C}_{nt} + \alpha_2^j (\bar{C}_{nt})^2 \\ \implies \Delta \ln(z_t^{nj}) &= \alpha_1^j \Delta \bar{C}_{nt} + \alpha_2^j \Delta (\bar{C}_{nt})^2 \end{aligned} \tag{13}$$

where z_t^{nj} is the productivity parameter in location n industry j, δ^{nj} is a location-industry specific constant, $\bar{C}_{nt}$ is the average annual temperature (in Celsius), and the coefficients α_1^j and α_2^j determine the quadratic impact of temperature on productivity. Given a path of temperature and estimates for α_1^j and α_2^j, equation (13) yields the change in productivity due to temperature rise.

We calibrate α_1^j and α_2^j based on the estimates of Cruz (2023), who uses annual production data from world regions to estimate the long-run effect of climate change on sectoral productivity.[19]

[19] Cruz (2023) estimates the damage functions for construction, trade and transportation, finance, and government

Table 1: Quadratic Impact of Temperature on Productivity

	Agriculture	Manufacturing	Service
α_1^j	2.998	1.339	1.292
	(1.268)	(1.344)	(1.051)
α_2^j	-0.143	-0.0615	-0.0485
	(0.0368)	(0.0472)	(0.0405)

Note: Estimates and standard errors (in parentheses) based on Cruz (2023).

Table 1 reports the estimates. Although the coefficients differ across sectors, all three sectors show an inverted U relationship between temperature and productivity. Importantly, these estimates are also quite uncertain, reflecting the inherent difficulty of inferring the economic toll of climate change. We will incorporate the uncertainty in these estimates into agents' beliefs.

Panel (a) of Figure 4 reports the paths of temperature under the medium carbon emissions scenario under RCP 4.5 for five broad U.S. regions: Northeast (NE), Southwest (SW), West (WE), Southeast (SE), and Midwest (MW). The solid lines present the median projection path for each region, which show an across-the-board increase in temperature. The dotted lines in the figure show the 5% and 95% percentiles of the temperature path for each region. The wide bands in the figure reflect that, even within the moderate carbon emissions scenario, the scientific community is fairly uncertain about how temperature will evolve. We can also see in the figure that the temperature is higher in the two southern regions (SW and SE) than in the rest of the country. Hence, given the inverted U relationship between temperature and productivity, even though the temperature rise is similar across the country, its negative impact will be larger in the South.

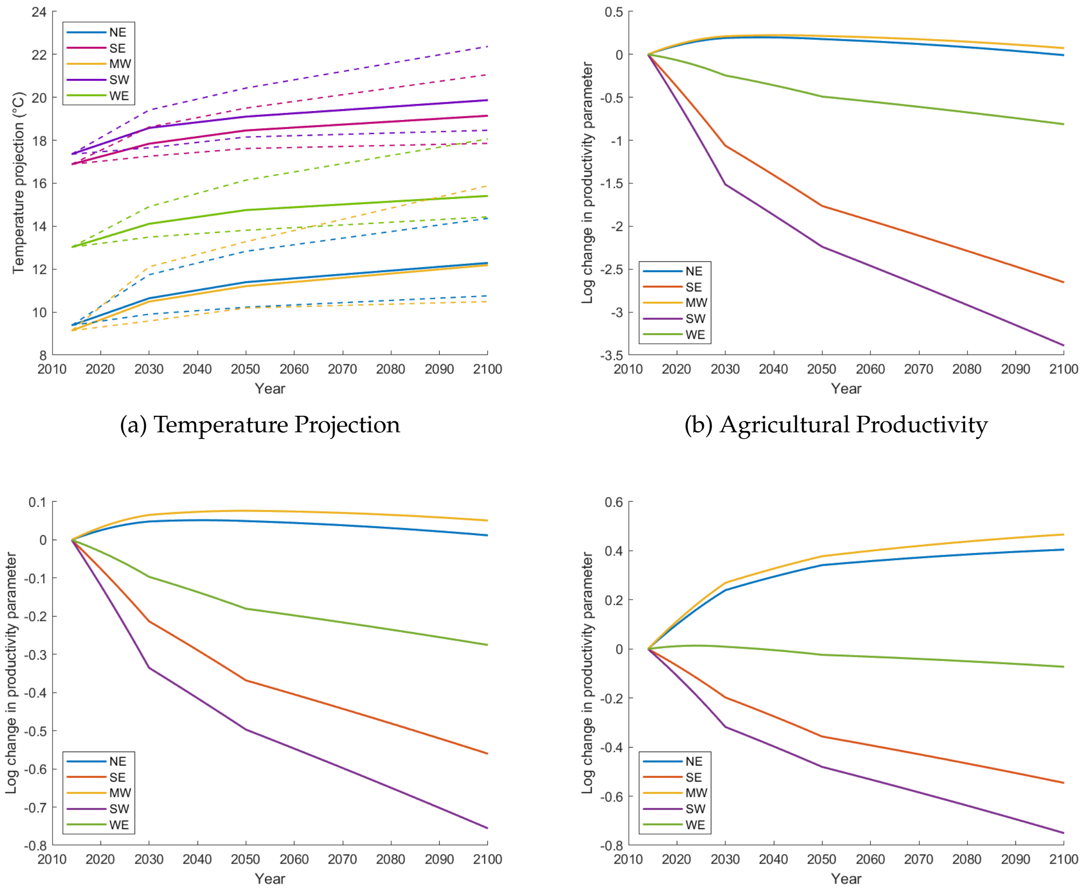
Panels (b), (c), and (d) of Figure 4 plot the population-weighted average of the change in $\ln(z_t^{ni})$ for U.S. regions calculated according to (13). The panels show that productivity decreases substantially in all three sectors in the South, but the decrease is particularly pronounced in agriculture, where $\ln(z_t^{nj})$ decreases by almost 300 log points.[20] On the other hand, the West sees only a modest decrease in productivity in each sector; the Northwest and Midwest see an increase in productivity. Although not shown in the figure, similar patterns hold for other countries; in general, high-latitude countries such as Canada and Russia experience an increase in productivity from climate change, whereas low-latitude countries such as India suffer productivity loss.

As the environment described above makes clear, in this application we take climate change (and its impact on the model fundamentals) as given and abstract from endogenous changes in carbon emissions that are considered in dynamic integrated models of climate (e.g., Nordhaus, 1994). The purpose of this simplification is to focus on the roles of uncertainty and heterogeneous beliefs. We note, however, that it is straightforward to extend our model to incorporate endogenous temperature rise.[21]

separately. We aggregate these coefficients based on their weight in the U.S. GDP to arrive at the coefficients for the broad service sector in our model. The productivity measure in Cruz (2023) is value-added productivity. We adjust these estimates by value-added shares to be consistent with our definition of z_{nj}^t as the TFP parameter.

[20]Recall that z_t^{nj} is the scale parameter of the Frechet distribution. In a closed economy, local TFP is $(z_t^{nj})^{\frac{1}{\theta}}$. In that case, a 300-log-points decrease in z maps to 66 log-points decreases in TFP when $\theta = 4.55$.

[21]In such an extension, by applying the propositions in the previous section, one can study how incorporating uncer-



(a) Temperature Projection

(b) Agricultural Productivity

(c) Manufacturing Productivity

(d) Service Productivity

Figure 4: Temperature Projection and Productivity Path

Note: Panel (a) shows the evolution of temperature for broad U.S. regions. The solid lines show the median of the temperature forecast, whereas the dotted lines represent the 5th and 95th percentiles of the forecast. Panels (b), (c), and (d) present the population-weighted change in the productivity parameters for broad U.S. regions under the median temperature path.

Constructing a deterministic path with perfect foresight. With the data on the initial allocation, we construct a deterministic path around which we will apply Propositions 3 and 4. To do so, we need to take a stand on the paths of all fundamentals underlying this path. We assume that on this path, local productivity evolves according to the damage function (13) and the median temperature projection under RCP 4.5, as plotted in Figure 4. We then obtain the deterministic path by solving a time-difference version of the model, as in Caliendo et al. (2019), assuming that all other fundamentals (trade and migration costs) are constant after 2014. Intuitively, the initial allocation contains information on the level of fundamentals in 2014 which, coupled with the time-change in fundamentals, gives us all the information necessary to solve the model. In constructing this time difference, we assume that the decisions of agents in 2014 were made under correct expectations about future fundamentals. We make this assumption so that we can compare our benchmark results on climate change with perfect foresight with those of the literature, which assumes that

tain, evolving, and heterogeneous beliefs about the impact of carbon emissions on temperature and productivity affects the economy.

agents have perfect foresight. However, it is worth noting that it is conceptually straightforward to allow the 2014 decision to be made under a different belief than the one implied by equation (13) and the median temperature projection.[22]

3.2 Climate Change vs. No-Change Scenario

To establish a benchmark result against which we can evaluate the importance of uncertainty and heterogeneous beliefs, we first compare the deterministic path with productivity evolving according to equation (13), constructed in the previous subsection, to a counterfactual deterministic path in which productivity does not change after 2014—the no-change scenario. The difference between these two paths delivers the economic effect of perfectly anticipated climate change.

The red dotted lines in Figure 5 show the evolution of the population across locations in the United States with perfectly anticipated climate change. As the temperature rises, the population of the South relocates elsewhere in the country. By 2100, the two southern regions together account for only 27% of the U.S. population, down from 37% at the beginning of the period. The blue solid lines in the figure show the evolution of the population across space under the 'no-change' scenario. We can see that if not for climate change, the population of the South would increase and the populations of other regions would decline.

Figure 6 plots the difference in welfare across states between the climate change scenario and the no-change scenario, with welfare measured as the change in consumption equivalents. Higher temperatures due to climate change benefit workers in northern states but hurt those in southern states. For instance, current residents of Alaska benefit by more than 20%, whereas those of Texas and Louisiana experience a 10% welfare loss. Overall, the population-weighted average welfare loss for the United States is 1.6%, broadly consistent with recent estimates using deterministic models (e.g., Cruz and Rossi-Hansberg, 2023, Cruz, 2023).

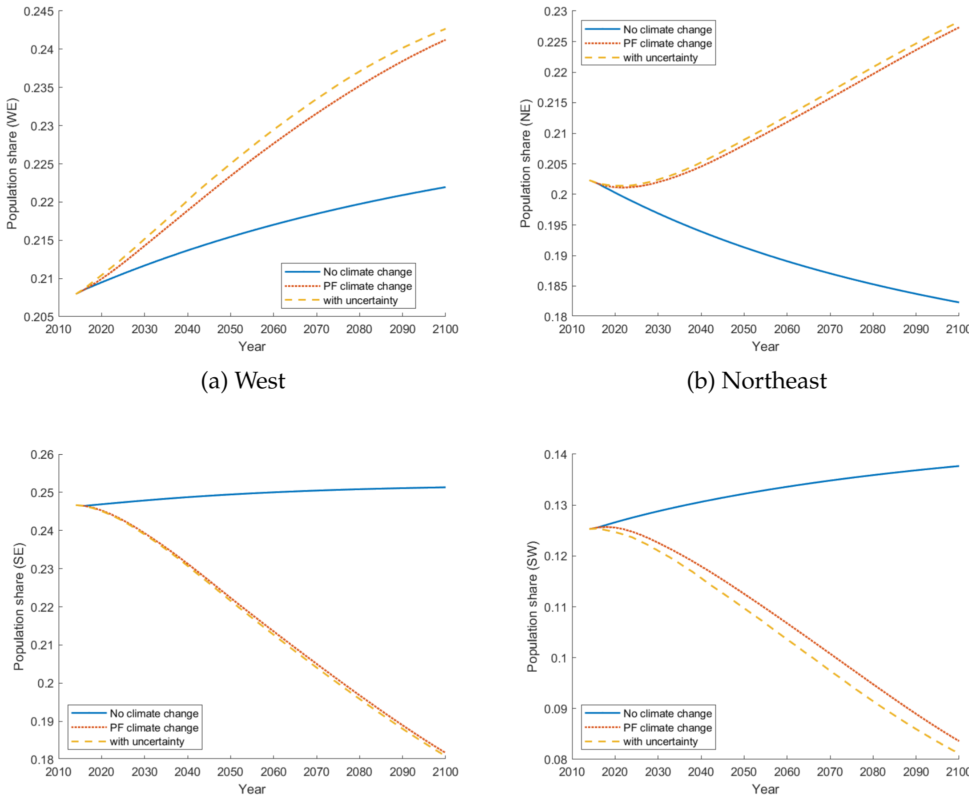
3.3 The Role of Uncertainty

We now turn to study the dynamic effects of climate change under uncertainty, which consists of both the uncertainty in the path of future temperature, shown in Panel (a) of Figure 4, and the uncertainty in how temperature affects productivity, shown in Table 1.

We apply Proposition 3 to study how such uncertainty affects welfare and population reallocation within the United States. Specifically, for each period $t = 1, .., T$, we implement the second-order approximation around the perfect foresight path by simulating S paths of future temperature rise for U.S. states and other countries. We solve the sequential equilibrium under each of the simulations and then use the moments calculated from these S solutions to solve for agents' decision in period t, which shifts the economy into the next period.

Before describing our findings, we discuss three additional aspects of this exercise. First, we

[22] In that case, we can first solve a time-difference version of the model to obtain the sequential equilibrium corresponding to agents' belief. We can then use Proposition 1 to obtain the deterministic path corresponding to fundamentals with median temperature change. In the same spirit, arbitrary time variations in trade and migration costs (and arbitrary beliefs about these) can also be incorporated.



(a) West (b) Northeast

(c) Southeast (d) Southwest

Figure 5: Climate Change and Spatial Reallocation in the United States

Note: The figure shows each U.S. region's share of the population under different scenarios. 'No climate change' refers to the scenario in which climate and productivity stay the same as in 2014; 'PF climate change' refers to the scenario with perfectly anticipated climate change; and 'with uncertainty' refers to the scenario in which agents face uncertainty about the severity of future climate change.

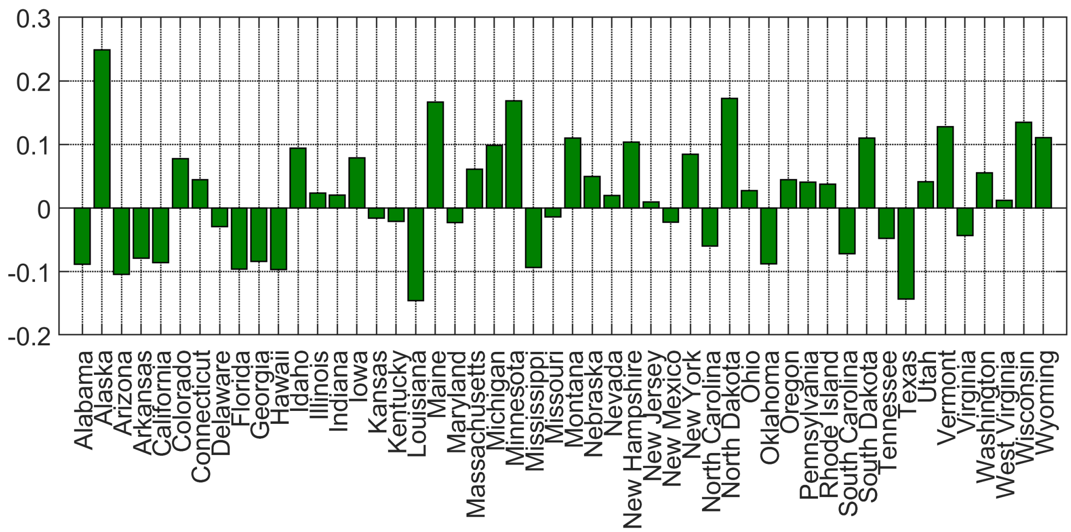


Figure 6: Welfare Effect of Perfectly Anticipated Climate Change Across U.S. States

Note: The figure plots the effect of climate change on welfare (in consumption equivalents) for residents in different states. We aggregate across sectors in each state using sectoral employment shares.

introduce uncertainty in temperature by adding mean-zero normal shocks around the median forecast for future temperatures. We calibrate the shock to match the dispersion between the percentiles of the future temperature distribution. Our assumption is essentially that agents *today* have similar information and belief as the scientific community on future temperature rise. In addition to specifying agents' beliefs today, to solve the model with uncertainty, we need to specify agents' beliefs in future periods under different realizations. The challenge is that while the belief of the scientific community today can be measured, how such beliefs will be revised in the future as new data arrive is necessarily speculative. For example, if in the year 2030, the realization of temperature is higher than the median forecast made today for that year, will the future belief in 2030 be revised up or down? Will the perceived uncertainty in 2030 about future temperature be higher or lower? To make progress, we assume that in any period t, agents' belief about the distribution for the period $t' \geq t$ is a normal distribution with mean $\mathcal{C}^t_{nt'}$ and variance $(\sigma^t_{nt'})^2$. We further assume that $\mathcal{C}^t_{nt'}$ follows a mean-reverting process that converges to the median forecast of temperature in t' by the scientific community today and that the uncertainty in period t' temperature perceived by workers standing at $t \leq t'$ depends only on $t' - t$. Formally,

$$\mathcal{C}^t_{nt'+1} - \bar{\mathcal{C}}_{nt'+1} = \rho(\mathcal{C}^t_{nt'} - \bar{\mathcal{C}}_{nt'}) \text{ and } \sigma^t_{nt'} = \sigma^1_{nt'-t+1}. \tag{14}$$

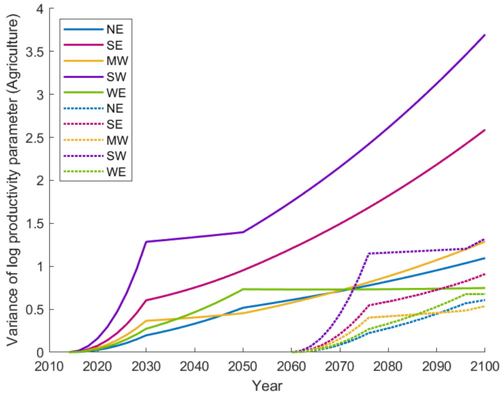
The term $\bar{\mathcal{C}}_{nt'}$ is the median forecast for the period t' temperature from $t = 1$. This setup implies that if the realization of $\mathcal{C}_{nt}$ is above $\bar{\mathcal{C}}_{nt}$, then agents think in future t', the average temperature will be higher than but eventually converge to $\bar{\mathcal{C}}_{nt'}$. We set $\rho = 0.9$, which implies a moderate convergence speed. We calibrate $\sigma^1_{nt'}$ to match the dispersion in the current belief about future temperature for each U.S. state and foreign regions displayed in Panel (a) of Figure 4.

Second, to capture the fact that climate change affects all states, in simulating future temperature paths, we assume that the percentiles of the draws for each state are perfectly correlated. For example, if Colorado has a draw for a period that is the 99th percentile according to its temperature distribution, then Utah also receives a 99th percentile draw from its distribution.

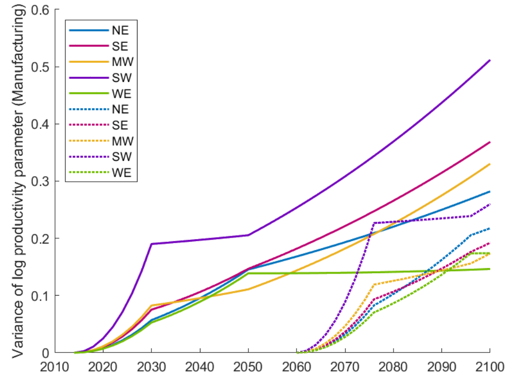
Third, as discussed before, in principle we can solve for the stochastic sequential equilibrium under any realization of future temperature rise. We do so for a particular realization: the median temperature path for which we have solved the deterministic sequential equilibrium. This choice ensures that as uncertainty converges to zero, the model with uncertainty will deliver the same allocation as the model with deterministic fundamentals, thereby isolating the role of uncertainty.

We obtain the deviation in the productivity path from the median temperature scenario by evaluating the simulated temperature path described above using equation (13). To take into account parameter uncertainty, in each simulation, we draw α^j_1 and α^j_2 from their asymptotic distributions according to the standard errors of the estimates.

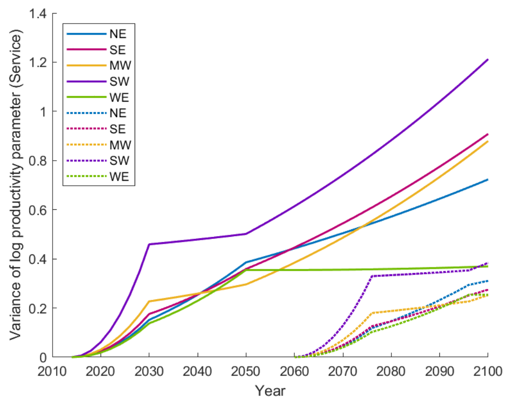
Figure 7 shows the variance across simulations of the deviations in log productivity from the deterministic path with climate change. As an illustration, the solid lines show the variance of the simulations from the year 2014, and the dotted lines show the variance of the simulations from the year 2060. Two observations emerge. First, the variance gradually increases, reflecting the fanning-out pattern in Panel (a) of Figure 4. Second, even though future temperature rises are



(a) Agriculture



(b) Manufacturing



(c) Service

Figure 7: Uncertainty About Future Productivity

Note: Each panel plots the variance of log productivity deviations from the perfect foresight path in a sector. The solid lines represent simulations from 2014; the dotted lines indicate simulations from 2060.

similarly uncertain across locations according to Figure 4, the variance in future productivity is much higher in the South and in agriculture than in other regions and other sectors. This spatial and sectoral heterogeneity arises because the higher temperature in the South and the damage function in agriculture imply a larger marginal effect of temperature rise on productivity in the South and in agriculture than in other regions and sectors. Thus, when the uncertainty in temperature/parameters is introduced, the variance of the change in productivity increases more there.

Turning to the results, the dashed lines in Figure 5 show the evolution of the population across regions with uncertainty. The population moves away from southern locations, and especially the Southwest, at a faster rate than in the case of deterministic fundamentals. This finding reflects the larger uncertainty about the productivity path that the South faces, which makes the region less attractive to risk-averse individuals than it is in a deterministic world. Overall, the Southwest experiences an additional 0.25-percentage-point decline in population due to uncertainty, adding to the 4.17-percentage-point decline in the scenario with deterministic fundamentals.

Considering uncertainty also affects welfare. Figure 8 plots the change in workers' welfare

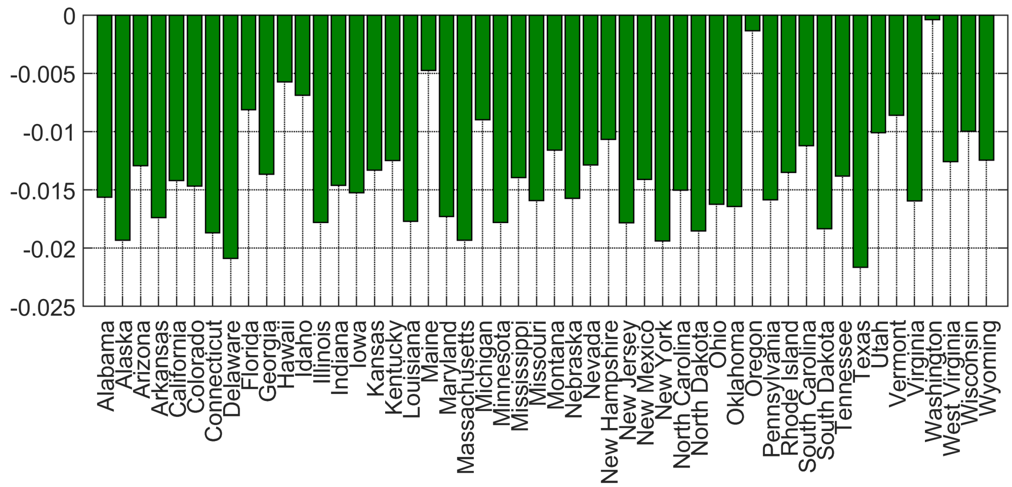


Figure 8: Welfare Effect of Uncertainty

Note: The figure plots the effect of introducing uncertainty in climate change (compared to perfectly anticipated climate change) on welfare (in consumption equivalents) based on agents' current state.

due to the introduction of uncertainty into the deterministic climate change scenario. Since individuals are risk-averse, their welfare decreases everywhere. The welfare decline tends to be more pronounced in the southwest and smaller in the west regions, which coincide with the size of the uncertainty about the productivity path in each region. For example, while the welfare in Texas decreases by 2.2%, the welfare loss in Washington is 0.03%. Aggregate welfare loss due to the uncertainty in climate change is about 1.48 percentage points. Adding this to the 1.6% welfare loss from perfectly anticipated climate change brings the welfare loss from climate change to 3.1%.

3.4 The Role of Disagreement

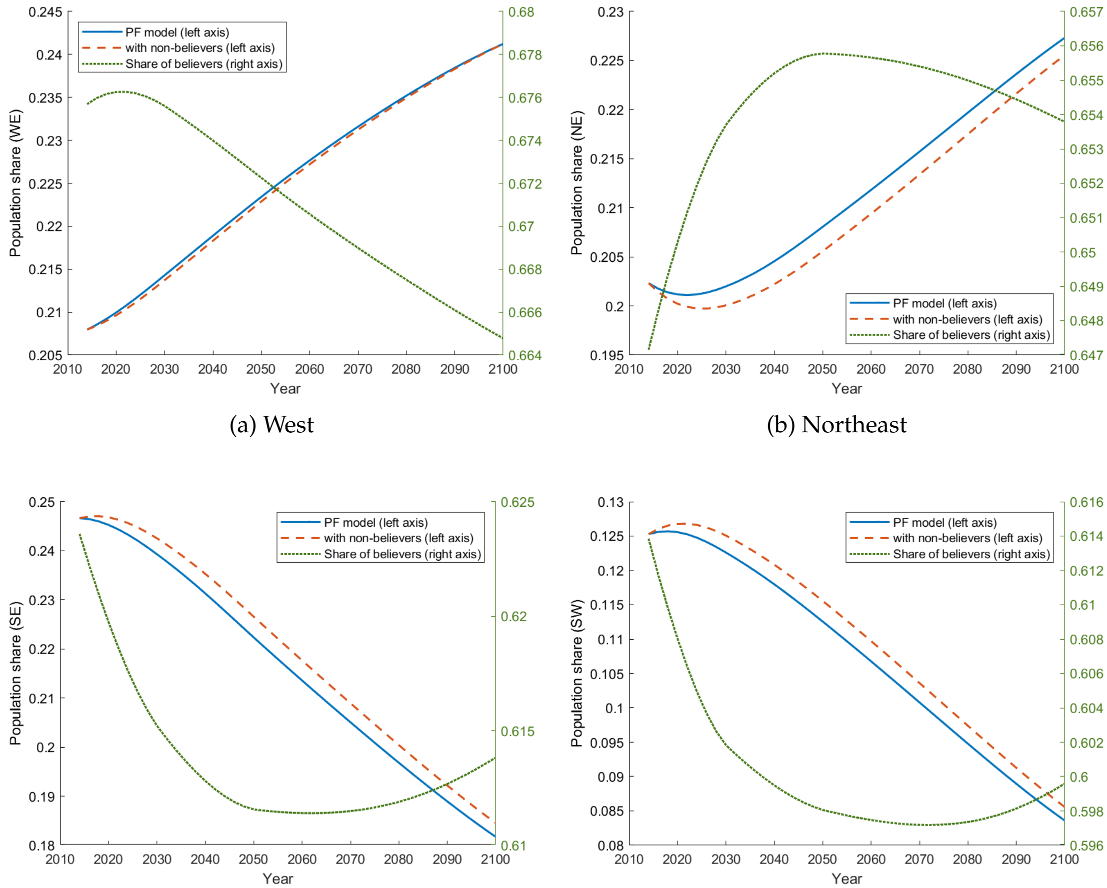
We now consider another departure from perfect foresight—heterogeneous beliefs among agents.

In Figure 1b, we illustrated the spatial heterogeneity in beliefs about climate change in 2021.[23] We incorporate such heterogeneous beliefs by assuming that in 2014, an exogenous share of the population in each state does not believe in climate change and thinks the future temperature will be the same as that in 2014. The remaining people believe in climate change and hold the belief that future productivity will evolve according to the median temperature path. We measure these shares using the 2014 vintage of the survey underlying Figure 1b. We also assume there is no uncertainty for either the believers or the skeptics.

We apply Proposition 4 to solve for the dynamic allocations with heterogeneous beliefs. In particular, we solve for the counterfactual path of this economy as deviations from the perfect foresight baseline scenario (where everyone is a believer) under Assumption 1, where each type thinks that the other type will be convinced by the data in the next period.

Figure 9 plots the reallocation effect in the presence of climate skeptics in red dashed lines (the left axis). Compared to the baseline scenario in which everyone believes in climate change, here,

[23] This disagreement might be correlated with other observables, such as age or skill, which our model abstracts from. Our methodology works similarly if such individual-level heterogeneity is considered.



(a) West (b) Northeast

(c) Southeast (d) Southwest

Figure 9: Reallocation Across Space in the Presence of Non-believers
Note: The left axes show the evolution of the population shares under perfect foresight (solid blue lines) and in the presence of non-believers (dashed lines). The right axes display the evolution of the share of believers across locations (dotted lines).

the presence of climate skeptics results in a higher share of the population located in the South and a slower relocation of the population across space. Intuitively, as a fraction of the population does not believe the temperature will rise, these people tend to remain in, or move to, the South. Consistent with this intuition, the green dotted lines (the right axis) show that generally, the share of believers decreases in the Southeast and Southwest and increases in the Northeast.

Figure 10 plots the change in the welfare of believers in the presence of climate skeptics relative to the baseline scenario. We find that believers in the North and Midwest tend to be better off, whereas believers currently in the South tend to be worse off. The increase in the welfare of believers in northern locations stems from the more muted migration to the North, which reduces the labor market competition faced by workers there; the same force leads to a decrease in the welfare of believers in the South. The change in welfare is noticeable for some states; for example, the change is 1.6 percentage points in Maine and 0.5 percentage points in Wisconsin. In the aggregate, the welfare of believers increases by 0.02 percentage points.

In summary, in this section, we have applied our methodology to study the impact of cli-

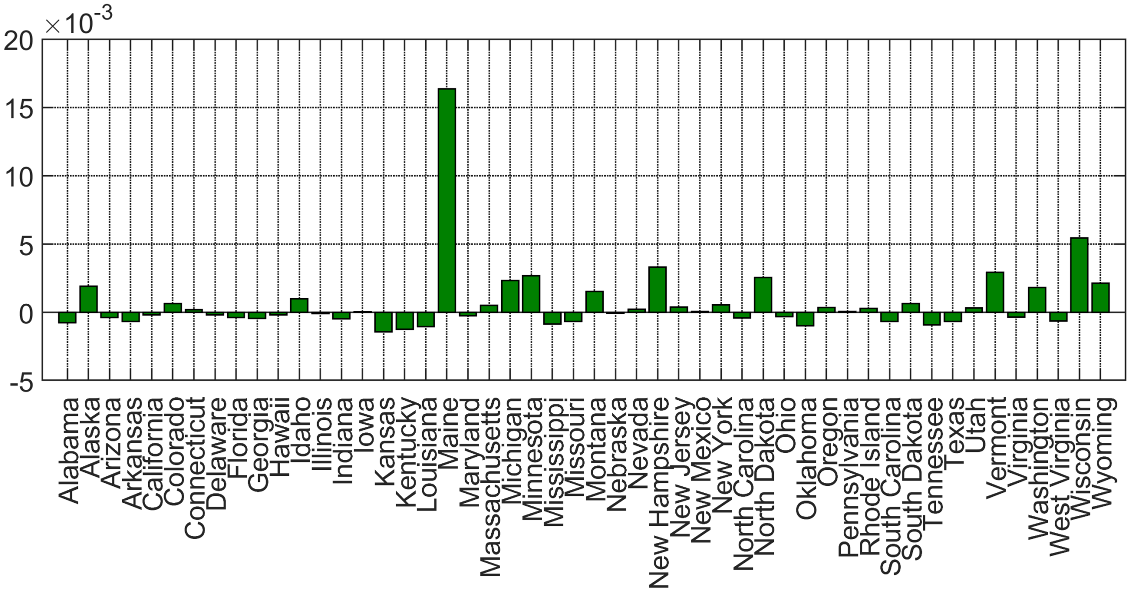


Figure 10: Welfare of Believers Compared to Perfect Foresight

Note: This figure plots the effect of introducing climate skeptics on the welfare (in consumption equivalents) of the believers of climate change based on the current state of believers.

mate change on welfare and population relocation across space within the United States under uncertainty and heterogeneous beliefs about the path of climate change. We find that introducing uncertainty leads to larger welfare losses in the United States and faster spatial reallocation of the population to the North and Midwest. On the other hand, the presence of climate skeptics reduces the welfare of believers in the South, increases the welfare of believers in the North, and slows down the spatial relocation.

4 Ex-post Application: The China Shock

Our second application considers the spatial effects of ex-post changes in fundamental productivity in the presence of evolving (and possibly biased) beliefs. We apply our methodology to study the impact of rapid productivity growth in China on the manufacturing employment share and welfare of workers in the United States.

As is well known, after China's accession to the World Trade Organization (WTO) in 2001, its exports to the United States skyrocketed. The rapid growth in Chinese exports might have been surprising to many observers in the United States, especially since China had received most-favored-nation treatment before 2001 (Handley and Limão, 2017). We model the source of import growth as driven by the catch-up in China's fundamental productivity with that of the United States. We allow workers in the United States to form beliefs ($f(z_{t+1}|z^t)$) about the catch-up process ($g(z_{t+1}|z^t)$) and will recover these beliefs by matching the employment projection at each moment in time. The belief processes and the actual allocation over the 'China shock' period allow us to recover the perfect foresight path as described in Section 2.5 and the counterfactual path without the China shock using Proposition 1.

4.1 Setup and Data

As in the climate change application, we extend the baseline model to incorporate multiple sectors and intermediate goods. Since imports from China are mostly manufacturing goods, we incorporate two sectors—manufacturing and others—in the model. Our analysis includes states in the United States and 28 other countries. We allow for migration across sectors and states within the United States, but not internationally. Following many existing quantitative studies of the China shock, we assume the flow utility $U(c_{it}) = \ln(c_{it})$.[24]

We assume that agents have perfect foresight about all fundamentals except for China's productivity, the premise being that not many people in 2001 anticipated China to catch up as quickly as it did.[25] We use the projections in Figure 1a to discipline these beliefs, implicitly assuming that if people had perfectly anticipated the productivity path of China, their projections about manufacturing employment shares would have been aligned with the actual shares. This choice is primarily motivated by growing research that emphasizes the role of trade with China in the decline in U.S. manufacturing activity (Autor et al., 2013; Pierce and Schott, 2016; Handley and Limão, 2017) over this period, but it is also consistent with the measured productivity dynamics between the two countries, which show the catch-up in productivity is mostly due to growth in China rather than abrupt changes in the United States, as we describe below.[26]

To recover the perfect foresight and counterfactual paths from the actual allocation, we start by constructing the time series of the actual migration, trade, and production data. As most of these data sources have been described in Section 3.1, here we describe only those additional steps that are necessary to construct the time series. First, the CFS covers only 2002, 2007, 2012, and 2017. We interpolate inter-state manufacturing trade between 2000 and 2016 using the information from the nearest covered year. Second, the job-to-job flow data that we use to construct the migration matrix misses some states as either the destination or the origin over the period 2000–2009, when the dataset gradually expanded to cover all states. We fill in the missing values by scaling future observations of each cell backward using the change in migration over time for that cell constructed by Caliendo et al. (2019). Third, to inform the belief process, we measure the productivity of China and other countries before the year 2000 using time series trade shares and price data. We obtain price data from the Penn World Table. Since the WIOD does not extend long enough before 2000, we obtain trade flows from the Eora Database.

4.2 Productivity Catch-up and Evolving Beliefs

We first use model inversion to recover the time series of manufacturing productivity in China and in the United States. In our model, domestic expenditure shares, wages, and price indices

[24] We set annual discount factor $\beta = 0.96$ and use the same migration and trade elasticities as in the first application.

[25] The very use of the term 'China shock' in the trade literature is consistent with this premise.

[26] Alternatively, we can assume that people expected U.S. manufacturing productivity to grow faster than it did. We view this alternative as less appealing given the lack of trend breaks in the U.S. productivity series.

satisfy the following relationship:

$$\lambda_t^{nj,nj} = z_t^{nj} \left(w_t^{nj} / P_t^{nj} \right)^{-\theta \gamma^{nj}},$$

where γ^{nj} are the value-added shares in production, w_t^{nj} is the wage, and P_t^{nj} is the price index from the source described previously. We can therefore measure z_t^{nj} using the data at hand.

The left panel of Figure 11 plots $\ln(z_t^{CHN,M}) - \ln(z_t^{USA,M})$ over time. We can see from the figure that in the first 10 years of the sample, China experienced some catch-up in relative productivity and then gave away part of the progress at the end of the 1990s. However, starting in 2000–2001, the year of China's accession to the WTO, the catch-up process accelerated. By 2016 the log difference in fundamental productivity with the United States shrank to approximately 0.5, which is a quarter of the log difference in the year 2000.

Motivated by the S-shaped pattern in relative productivity after 2000, we model the catch-up process as a logistic function given by

$$\ln(z_t^{CN,M}) - \ln(z_t^{US,M}) = \alpha_3 \cdot \frac{\exp\left(\alpha_1(t-1999)\right)}{\exp\left(\alpha_1(t-1999)\right) + \alpha_2}, \tag{15}$$

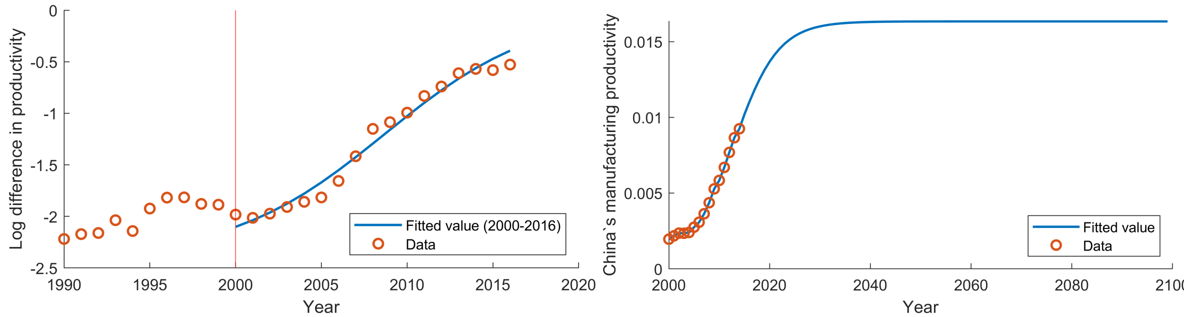
where the coefficients α_1, α_2, and α_3 govern the catch-up speed, initial productivity gap, and average productivity of China relative to the United States, respectively. We estimate this specification using data over the period 2000–2016. We estimate that $\alpha_1 = -0.22$, $\alpha_2 = 0.11$, and $\alpha_3 = -2.4$. The solid line in the left panel of Figure 11 shows the fit. While the estimated model implies a smoother catch-up process than the data, overall it fits the data well.

We treat the estimated empirical catch-up model as the true process governing the manufacturing productivity of China relative to that of the United States. Assuming that the U.S. productivity is constant after 2016 and applying the estimated process, we obtain the level of manufacturing productivity in China, which is shown in the right panel of Figure 11. Under this specification, China's productivity will grow until 2040, when it catches up with that of the United States. This solid line is essentially the $g(z_{t+1}|z^t)$ in the theoretical section.

Forward-looking agents form beliefs about the catch-up speed, captured by α_1, and update their beliefs as new data arrive. We recover agents' belief about α_1—and by implication, their belief about the fundamental productivity in China, $f(z_{t+1}|z^t)$—for the period 2001–2006 by implementing the following algorithm:

1. Choose agents' beliefs about α_1 in each of the following years—2002, 2004, and 2006—interpolating the beliefs for the years in between.[27]

[27] We choose the belief parameter for only these even-number years because employment projections from the BLS are available for only these years. We assume that starting from 2008, agents in the economy have perfect foresight. This assumption is motivated by the observation that in Figure 1a, by 2006-2008, which reflects the end of the China shock period, the employment projection is already largely aligned with the data. We note that our methodology does not require that agents have perfect foresight when the data ends. We can always first solve for a sequential path according to agents' belief from the end of the data, from which it is straightforward to recover the perfect foresight path, as we discussed in detail in the theory section.



(a) CHN–US Log Productivity Difference

(b) Fitted Productivity for China

Figure 11: Measured Manufacturing Productivity in United States and China

Note: The left panel plots the log difference between manufacturing productivity in China and the United States from 1990 to 2016 and the fitted productivity based on data from 2000 to 2016. The right panel plots the level of China's manufacturing productivity according to the fitted model in and out of the sample period.

2. Given beliefs about α_1, generate beliefs about China's productivity using equation (15).

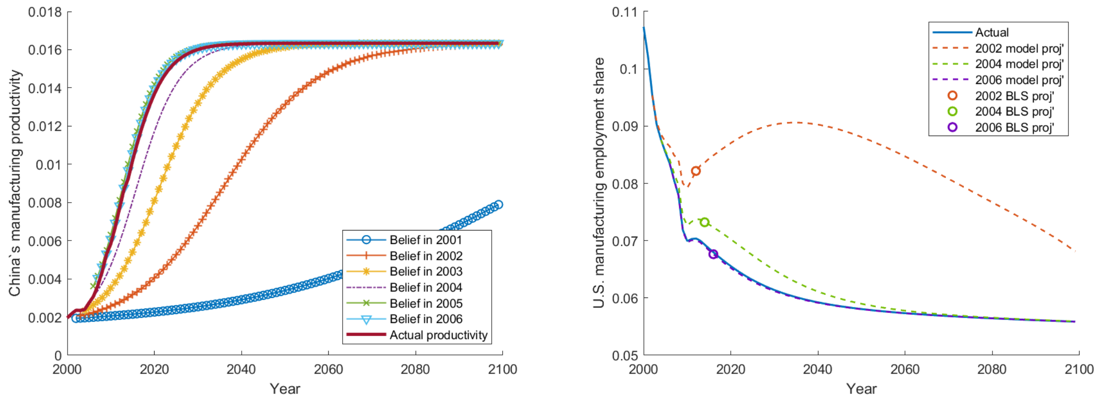
3. Use the algorithm described in Section 2.5 backward from 2007 to 2001, along the way recovering agents' beliefs about the manufacturing employment share in each year.

4. Check whether the model-implied expected future manufacturing employment shares match the empirical projection; if not, return to step 1 and revise the belief parameters.

Figure 12 illustrates the outcomes from this algorithm. The left panel presents the recovered beliefs about the level of China's manufacturing productivity. Each color represents the belief formed in a particular year. The solid line is the actual process for productivity. In the initial years, agents were too pessimistic about China's productivity. Gradually, they revised their belief, and by 2006, the belief was almost fully aligned with actual productivity. The right panel shows that under this belief, the model-implied manufacturing employment share projection (dashed lines) is consistent with the BLS 10-year projection represented by circles.

4.3 The Impact of the China Shock Under Evolving Beliefs

With the belief process described in the previous subsection, we apply the methodology described in Section 2.5 to study the impact of the China shock. We think of the 'shock' as the rapid productivity catch-up of China that might have been surprising to agents. Accordingly, we define the counterfactual no-shock scenario as one in which China's manufacturing productivity grows at a slower rate (e.g., 3.5% yearly) until reaching its steady-state value, and in which agents perfectly foresee this slower catch-up. We answer two questions: first, how would the U.S. economy have evolved with a slower (and perfectly anticipated) China productivity catch-up? The difference between this counterfactual path and the actual data reflects the impact of faster productivity growth in China and agents' evolving beliefs about this growth on the evolution of the economy. Since the bulk of the literature using dynamic spatial models has assumed that the data are generated by agents with perfect foresight, we pose a second question: How differently would the counterfactual economy have evolved if we had imposed that in the data agents have perfect foresight?



(a) Belief About China's Productivity (b) Manufacturing Employment Share Projection

Figure 12: Agents' Beliefs and Manufacturing Employment

Note: The left panel presents the actual productivity process of China's productivity (red solid line) and the recovered beliefs about this process formed at different years (all others). The right panel presents the actual manufacturing employment share (blue solid line); the model-implied manufacturing employment share projections starting from 2002, 2004, and 2006 (dashed lines); and the BLS 10-year projections (circles) from the corresponding years.

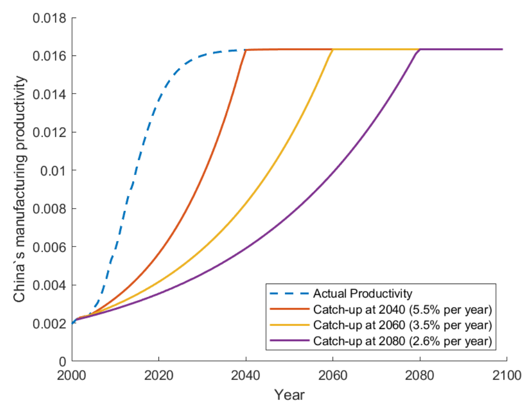
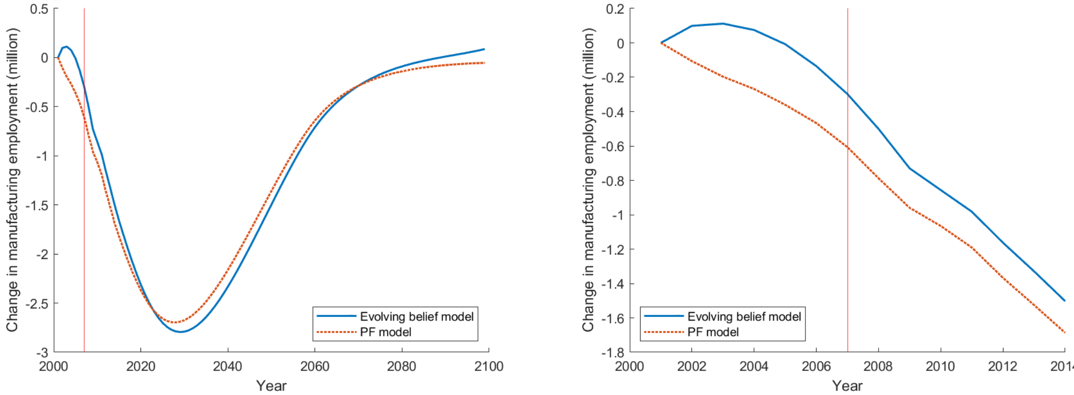


Figure 13: Counterfactual Productivity Path for China

Note: This figure presents the actual productivity process of China's manufacturing productivity (blue dashed line) and three counterfactual productivity processes corresponding to different growth rates (solid lines).

Figure 13 compares the counterfactual productivity processes under different assumptions on the growth rates with the actual productivity process. We use a 3.5% per annum growth rate, which implies that productivity in China will reach the steady state in 2060, about 30 years later than implied by the true catch-up process.

The blue solid line in Figure 14 shows the decline in manufacturing employment in the model with evolving beliefs relative to a counterfactual scenario with slower and perfectly anticipated catch-up of China's productivity. The red dotted line displays the inferred decline if we had assumed that the data are generated by agents with perfect foresight. In both models, the decline intensifies even after 2014 (the end of sample) as the difference between the actual (dashed line in Figure 13) and counterfactual productivity paths of China (3.5% growth scenario in Figure 13) increases over time. The employment decline eventually reverses as both the actual and the coun-



(a) 2001-2099

(b) In-sample Period (2001–2014)

Figure 14: Employment Effect of the China Shock

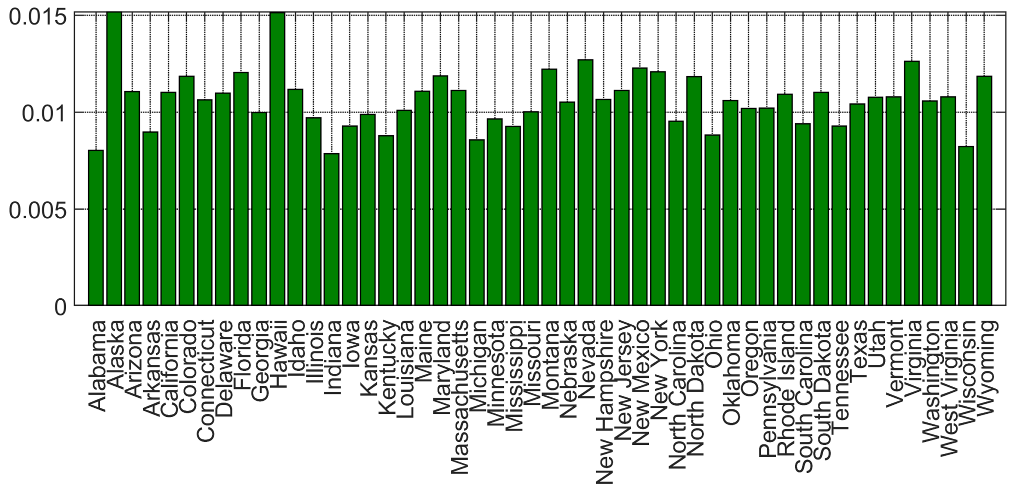
Note: The left panel presents the effects of the China shock measured as the change in manufacturing employment between the economy with the China shock and the economy without the China shock (China's manufacturing productivity grows at 3.5% per annum) under the perfect foresight model (PF model) and the benchmark model with evolving beliefs. The right panel focuses on the data period: from 2001 to 2014.

terfactual productivity converge to the same level.

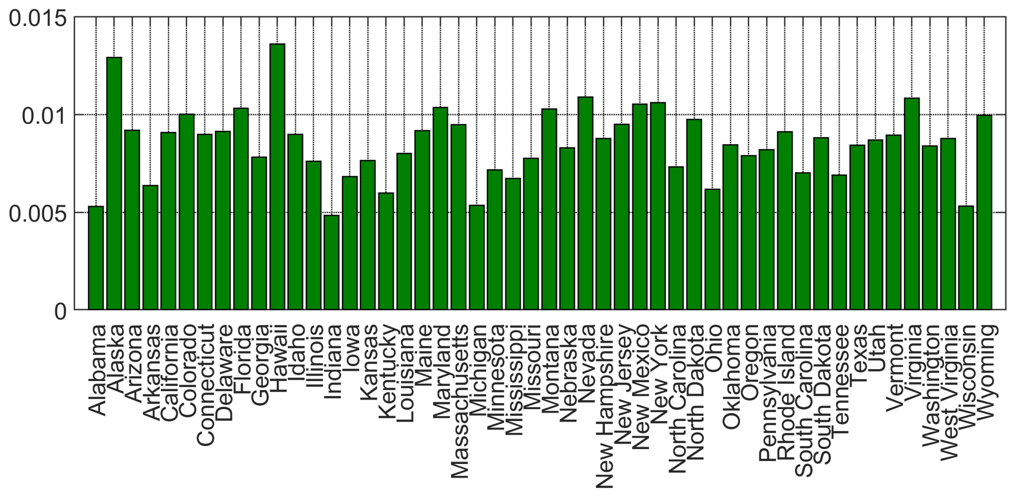
To answer our second counterfactual question, in the right panel, we can see the perfect foresight model infers immediate reallocation out of manufacturing as workers perfectly anticipate productivity growth. On the other hand, with evolving beliefs, agents initially do not anticipate productivity growth in China, so they do not reallocate from manufacturing. Subsequently, as agents update their beliefs (recall that agents' beliefs became aligned with the actual evolution by 2006–2008), manufacturing employment declines and the gap between this decline and the manufacturing decline under perfect foresight shrinks. However, even by 2014, the difference in the inferred decline of manufacturing employment between evolving beliefs and perfect foresight models remains more than 10%.

Figure 15 reports the welfare effect of the China shock on U.S. states with evolving beliefs and with perfect foresight. On average, the welfare gains under evolving beliefs (1.11%) are around one-third higher than they are under perfect foresight (0.81%). The difference in the inferred welfare gains is heterogeneous across space. For example, for Alabama, the model with evolving beliefs infers welfare gains that are about one-half higher than under perfect foresight. One might have the intuition that perfect foresight enables agents to make more informed decisions, thereby improving their welfare, but it is important to note that this exercise is not about the value of information. Instead, it is about how researchers making different assumptions will infer different welfare effects. In the stochastic model with evolving beliefs, observed decisions are made under overly pessimistic beliefs about China's productivity, so it implicitly rationalizes the actual migration rates with lower mobility costs than those of the perfect foresight environment, which in turn leads to higher welfare gains from the China shock.[28] Evolving beliefs also imply a slower

[28]In Appendix B.3.3, we measure the importance of incorporating evolving beliefs when explaining the manufacturing share in terms of migration costs. In particular, we multiply the manufacturing-to-non-manufacturing migration costs by $(1+x)$ over 2001-2007 from the perfect foresight path *recovered by the model with evolving beliefs*. We choose



(a) Model with Evolving Beliefs



(b) Model with Perfect Foresight

Figure 15: Welfare Effect of the China Shock

Note: The top figure presents the change in welfare (in consumption equivalents) in each state in the model with evolving beliefs, and the bottom figure presents the change in welfare in each state under perfect foresight.

transition out of manufacturing than the perfect foresight model, as we discussed before. The decrease in manufacturing imports under this slower manufacturing employment reallocation leads to improvements in the U.S. terms of trade, further increasing welfare.

5 Conclusion

Agents' anticipation about the future path of fundamentals plays an important role in shaping the impact of shocks in dynamic spatial environments. In this paper, we have built a generic dynamic spatial equilibrium model with stochastic fundamentals. We develop a local approximation

x so that the new path has the same manufacturing employment in 2007 as in the data. We find $x \approx 0.5$, suggesting that to account for the observed path of manufacturing employment share, the model with evolving beliefs needs approximately one-third lower costs for switching out from the manufacturing sector than the perfect foresight model.

method to solve this model for counterfactual analysis, allowing beliefs about these fundamentals to be evolving, uncertain, and heterogeneous among groups of agents. In scenarios in which the researchers observe a sequence of data generated by an economy with imperfect beliefs and other time-varying fundamentals but do not observe the fundamentals directly, we show how to disentangle the roles of beliefs and changes in model fundamentals in shaping economic outcomes.

To showcase our methodology, we apply it to two settings: an ex-ante study of the spatial reallocation in response to climate change and an ex-post study of the impact of China's manufacturing productivity growth on U.S. employment. Both quantitative applications demonstrate the importance of incorporating the departures in agents' beliefs from perfect foresight.

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Online Appendix: Learning and Expectations in Dynamic Spatial Economies

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Appendix A Theory

This appendix presents the proofs of the propositions and additional derivations. Throughout, we define $\hat{v}_{it+1}(z^t) \equiv v_{it+1}(z^t) - \bar{v}_{it+1}$ and $\hat{v}_{it}(z^t) \equiv v_{it}(z^t) - \bar{v}_{it}$. For all other exogenous or endogenous variables x, we define $\hat{x}_t(z^t) \equiv \ln \hat{x}_t(z^t) - \ln \bar{x}_t$. To simplify the notation, we denote a conditional expectation by $\mathbb{E}_t x_{nt'} \equiv \mathbb{E}[x_{nt'}(z^{t'})|z^t]$. The proofs and additional derivations in this appendix are based on the one-sector version of the model; the algebra for the multi-sector version of the model used in the quantification sections is relegated to a supplementary appendix.

We maintain that trade and migration costs are perfectly anticipated in our quantitative applications. To demonstrate that our methodology can accommodate other fundamentals besides productivity being stochastic, in all proofs and derivations provided in this appendix, we allow trade costs κ_{nit} to also be stochastic. Stochastic migration costs can be incorporated analogously.

A.1 Proof of Proposition 1

We prove the proposition by deriving the first-order approximation of equations (2)-(7), with the detailed algebra relegated to Supplementary Appendix C.1.

The outcomes in period t and the expected outcomes for all future periods $t' > t$ from the perspective of agents in t satisfy the following system of equations:

For period t,

$$\hat{v}_{nt}(z^t) = \frac{\partial U(\bar{w}_{nt}, \bar{P}_{nt})}{\partial \ln \bar{w}_{nt}} \hat{w}_{nt}(z^t) + \frac{\partial U(\bar{w}_{nt}, \bar{P}_{nt})}{\partial \ln \bar{P}_{nt}} \hat{P}_{nt}(z^t) + \beta \sum_{i=1}^{N} \bar{\mu}_{nit} \mathbb{E}_t \hat{v}_{it+1} \quad \text{(A.1)}$$

$$\hat{\mu}_{nit}(z^t) = \frac{\beta}{\nu} \sum_{m=1}^{N} \Big(\mathbb{1}(m=i) - \bar{\mu}_{nmt} \Big) \mathbb{E}_t \hat{v}_{mt+1} \quad \text{(A.2)}$$

$$\hat{l}_{nt+1}(z^t) = \sum_{i=1}^{N} \frac{\bar{\mu}_{int} \bar{l}_{it}}{\bar{l}_{nt+1}} \Big(\hat{\mu}_{int}(z^t) + \hat{l}_{it}(z^{t-1}) \Big) \quad \text{(A.3)}$$

$$\hat{\lambda}_{nit}(z^t) = -\theta \left(\hat{w}_{it}(z^t) + \hat{\kappa}_{nit} - \hat{P}_{nt}(z^t) \right) + \hat{z}_{it} \quad \text{(A.4)}$$

$$\hat{P}_{nt}(z^t) = \sum_{i=1}^{N} \bar{\lambda}_{nit} \left(\hat{w}_{it}(z^t) + \hat{\kappa}_{nit} - \frac{1}{\theta} \hat{z}_{it} \right) \quad \text{(A.5)}$$

$$\hat{w}_{nt}(z^t) + \hat{l}_{nt}(z^{t-1}) = \sum_{i=1}^{N} \frac{\bar{\lambda}_{int} \bar{w}_{it} \bar{l}_{it}}{\bar{w}_{nt} \bar{l}_{nt}} \Big(\hat{\lambda}_{int}(z^t) + \hat{w}_{it}(z^t) + \hat{l}_{it}(z^{t-1}) \Big) \quad \text{(A.6)}$$

For periods t'>t,

$$\mathbb{E}_t \hat{v}_{nt'} = \frac{\partial U(\bar{w}_{nt'}, \bar{P}_{nt'})}{\partial \ln \bar{w}_{nt'}} \mathbb{E}_t \hat{w}_{nt'} + \frac{\partial U(\bar{w}_{nt'}, \bar{P}_{nt'})}{\partial \ln \bar{P}_{nt'}} \mathbb{E}_t \hat{P}_{nt'} + \beta \sum_{i=1}^{N} \bar{\mu}_{nit'} \mathbb{E}_t \hat{v}_{it'+1} \quad \text{(A.7)}$$

$$\mathbb{E}_t \hat{\mu}_{nit'} = \frac{\beta}{\nu} \sum_{m=1}^{N} \Big(\mathbb{1}(m=i) - \bar{\mu}_{nmt'} \Big) \mathbb{E}_t \hat{v}_{mt'+1} \quad \text{(A.8)}$$

$$\mathbb{E}_t \hat{l}_{nt'+1} = \sum_{i=1}^{N} \frac{\bar{\mu}_{int'} \bar{l}_{it'}}{\bar{l}_{nt'+1}} \left(\mathbb{E}_t \hat{\mu}_{int'} + \mathbb{E}_t \hat{l}_{it'} \right) \quad \text{(A.9)}$$

$$\mathbb{E}_t \hat{\lambda}_{nit'} = -\theta (\mathbb{E}_t \hat{w}_{it'} + \mathbb{E}_t \hat{\kappa}_{nit'} - \mathbb{E}_t \hat{P}_{nt'}) + \mathbb{E}_t \hat{z}_{it'} \quad \text{(A.10)}$$

$$\mathbb{E}_t \hat{P}_{nt'} = \sum_{i=1}^{N} \bar{\lambda}_{nit'} \left(\mathbb{E}_t \hat{w}_{it'} + \mathbb{E}_t \hat{\kappa}_{nit'} - \frac{1}{\theta} \mathbb{E}_t \hat{z}_{it'} \right) \quad \text{(A.11)}$$

$$\mathbb{E}_t \hat{w}_{nt'} + \mathbb{E}_t \hat{l}_{nt'} = \sum_{i=1}^{N} \frac{\bar{\lambda}_{int'} \bar{w}_{it'} \bar{l}_{it'}}{\bar{w}_{nt'} \bar{l}_{nt'}} \left(\mathbb{E}_t \hat{\lambda}_{int'} + \mathbb{E}_t \hat{w}_{it'} + \mathbb{E}_t \hat{l}_{it'} \right). \quad \text{(A.12)}$$

Solving this system of linear equations requires as inputs (i) outcomes along a deterministic path with perfect foresight, $\{\bar{\mu}_{nit'}, \bar{l}_{nt'}, \bar{\lambda}_{nit'}, \bar{w}_{nt'}, \bar{P}_{nt'}\}_{i=1,n=1,t'=t}^{N,N,T}$ as approximation points; (ii) the expectation of agents in period t on the path of future fundamentals as deviations from the fundamentals underlying the perfect foresight path $\mathbb{E}_t \hat{z}_{t'}$; and (iii) the distribution of labor in period t, $\hat{l}_{it}$. The system delivers as outputs agents' decision and equilibrium outcomes in period t, as well as agents' expectations about future decisions and equilibrium outcomes in all $t' > t$.

Algorithm. We solve this system of equations using a shooting algorithm. As detailed in the supplementary appendix, we iterate on the guesses of the path of expected location values according to agents' belief in period t ($\mathbb{E}_t \hat{v}_{nt'}$). Given a guess of these values, we use equations (A.2) and (A.8) to obtain the expected path of the population distribution, with which we solve for the static trade outcomes using equations (A.4)-(A.6) and (A.10)-(A.12). We construct the path of expected real wages using the solution and then update the guess on the path of expected location values using (A.1) and (A.7). We iterate on this process until convergence, which delivers agents' actual migration decisions at time t and therefore moves the economy to the next period. We then repeat the procedure to solve for the actual allocations at each time $t+1$, $t+2$,..., to obtain the evolution of the economy.

This process requires solving the static trade equilibrium numerous times. We provide in the supplementary appendix the analytical expressions of outcomes in the static trade equilibrium ($\mathbb{E}_t \hat{\lambda}_{nit'}, \mathbb{E}_t \hat{P}_{it'}, \mathbb{E}_t \hat{w}_{it'}$) for an extended model with multiple sectors and intermediate inputs, so solving the static trade equilibrium is a simple matrix inversion (see also Kleinman et al., 2023).

A.2 Proof of Proposition 2

In this subsection, we first explain the algorithm presented in Section 2.5 for recovering belief and perfect foresight paths from the observed allocation. We then describe how we use the belief paths and the actual allocation to measure welfare when agents do not have perfect foresight.

Recovering belief and perfect foresight paths. As in the main text, we use variables with a bar $\bar{}$ to denote the approximation points. However, as mentioned in the main text, the perfect foresight path is unobserved. Hence, we use the actual outcomes from the trade equilibrium and belief paths as approximation points. We denote the *actual* outcomes in period t by variables with a tilde $\tilde{}$, i.e., $\tilde{v}_{nt}$, $\tilde{w}_{nt}$, $\tilde{P}_{nt}$, $\tilde{\lambda}_{nit}$, $\tilde{\mu}_{nit}$, $\tilde{l}_{nt}$.

i. Starting from period T, we solve for the expected outcomes of the trade equilibrium in period T according to the beliefs of agents in period $T-1$, namely, $\mathbb{E}_{T-1} w_{iT}$, $\mathbb{E}_{T-1} P_{iT}$, and $\mathbb{E}_{T-1} \lambda_{inT}$ for all i, n. We do so by approximating around the actual outcomes in period T. Define $\mathbb{E}_{T-1} \hat{x}_T \equiv \mathbb{E}_{T-1} \ln(x_T) - \ln(\tilde{x}_T)$ for $x \in \{w, P, \lambda, l\}$. These hat variables satisfy the following system of linear equations:

$$\mathbb{E}_{T-1} \hat{\lambda}_{niT} = -\theta \Big(\mathbb{E}_{T-1} \hat{w}_{iT} + \mathbb{E}_{T-1} \hat{\kappa}_{niT} - \mathbb{E}_{T-1} \hat{P}_{nT} \Big) + \mathbb{E}_{T-1} \hat{z}_T \tag{A.13}$$

$$\mathbb{E}_{T-1} \hat{P}_{nT} = \sum_{i=1}^{N} \bar{\lambda}_{niT} \Big(\mathbb{E}_{T-1} \hat{w}_{iT} + \mathbb{E}_{T-1} \hat{\kappa}_{niT} - \frac{1}{\theta} \mathbb{E}_{T-1} \hat{z}_T \Big) \tag{A.14}$$

$$\mathbb{E}_{T-1} \hat{w}_{nT} + \mathbb{E}_{T-1} \hat{l}_{nT} = \sum_{i=1}^{N} \frac{\bar{\lambda}_{inT} \bar{w}_{iT} \bar{l}_{iT}}{\bar{w}_{nT} \bar{l}_{nT}} \Big(\mathbb{E}_{T-1} \hat{\lambda}_{inT} + \mathbb{E}_{T-1} \hat{w}_{iT} + \mathbb{E}_{T-1} \hat{l}_{iT} \Big) \tag{A.15}$$

where $\mathbb{E}_{T-1} \hat{l}_{iT} = 0$ (since the actual labor allocation in T is determined by the migration decision at $T-1$) and the approximation points are the actual outcomes in period T, namely, $(\bar{\lambda}_{inT}, \bar{w}_{nT}, \bar{P}_{nT}, \bar{l}_{nT}) \equiv (\tilde{\lambda}_{inT}, \tilde{w}_{nT}, \tilde{P}_{nT}, \tilde{l}_{nT})$.

The input for this system of equations is the actual outcomes in T and the deviations in agents' belief in period $T-1$ about the productivity in T from the actual productivity, namely, $\mathbb{E}_{T-1}\hat{z}_T$ (and in the case of stochastic trade costs, $\mathbb{E}_{T-1}\hat{\kappa}_{inT}$). The output of this system of equations is deviations such as $\mathbb{E}_{T-1}\hat{\lambda}_{niT}$, with which we can recover the level outcomes using the following relationship: $\mathbb{E}_{T-1}\ln(x_T) = \mathbb{E}_{T-1}\hat{x}_T + \ln(\tilde{x}_T)$ for $x \in \{w, P, \lambda, l\}$.

ii. Append the output from the first step by $\{\tilde{\mu}_{niT-1}, \tilde{w}_{nT-1}, \tilde{P}_{iT-1}, \tilde{\lambda}_{iT-1}, \tilde{l}_{nT-1}\}$, namely, agents' *actual* migration decision and the actual outcomes of the static trade equilibrium in period $T-1$. As $\{\tilde{\mu}_{niT-1}, \tilde{w}_{nT-1}, \tilde{P}_{iT-1}, \tilde{\lambda}_{iT-1}\}$ are all determined in period $T-1$, when the agents see the realization of productivity in $T-1$, together with the output from (i), they constitute the solution to the agents' problem in period $T-1$, as defined in Proposition 1.
From steps (i) and (ii), we obtain $\{\mathbb{E}_{T-1}\lambda_{nit}, \mathbb{E}_{T-1}P_{nt}, \mathbb{E}_{T-1}w_{nt}, \mathbb{E}_{T-1}l_{nt}, \mathbb{E}_{T-1}\mu_{nit}\}_{i=1,n=1,t=T-1}^{N,N,T}$, namely, the outcomes in $\{T-1, T\}$ according to agents' belief in period $T-1$.

iii. Derive the expected outcomes in periods $\{T-1, T\}$ according to agents' belief in period $T-2$ by approximating around the solution to agents' problem in period $T-1$ obtained previously. Doing so gives a system of linear equations (see below) with the input being (a) the outputs from steps (i) and (ii), and (b) the deviations in agents' beliefs in period $T-2$ about the productivity in $\{T-1, T\}$ from their beliefs in period $T-1$ about it, namely, $\mathbb{E}_{T-2}\ln(z_t) - \mathbb{E}_{T-1}\ln(z_t)$ for $t \in \{T-1, T\}$.

The output of the equations is $\{\mathbb{E}_{T-2}\hat{\lambda}_{nit}, \mathbb{E}_{T-2}\hat{P}_{nt}, \mathbb{E}_{T-2}\hat{w}_{nt}, \mathbb{E}_{T-2}\hat{l}_{nt}, \mathbb{E}_{T-2}\hat{\mu}_{nit}\}_{i=1,n=1,t=T-1}^{N,N,T}$, which can be used to recover the corresponding level variables, as explained previously.

The system of equations for $t \in \{T-1, T\}$ is given by

$$\begin{aligned}
\mathbb{E}_{T-2}\hat{\lambda}_{nit} &= -\theta\Big(\mathbb{E}_{T-2}\hat{w}_{it} + \mathbb{E}_{T-2}\hat{\kappa}_{nit} - \mathbb{E}_{T-2}\hat{P}_{nt}\Big) + \mathbb{E}_{T-2}\hat{z}_t \\
\mathbb{E}_{T-2}\hat{P}_{nt} &= \sum_{i=1}^{N}\bar{\lambda}_{nit}\Big(\mathbb{E}\hat{w}_{it} + \mathbb{E}_{T-2}\hat{\kappa}_{nit} - \frac{1}{\theta}\mathbb{E}_{T-2}\hat{z}_t\Big) \\
\mathbb{E}_{T-2}\hat{w}_{nt} + \mathbb{E}_{T-2}\hat{l}_{nt} &= \sum_{i=1}^{N}\frac{\bar{\lambda}_{int}\bar{w}_{it}\bar{l}_{it}}{\bar{w}_{nt}\bar{l}_{nt}}\Big(\mathbb{E}_{T-2}\hat{\lambda}_{int} + \mathbb{E}_{T-2}\hat{w}_{it} + \mathbb{E}_{T-2}\hat{l}_{it}\Big) \\
\mathbb{E}_{T-2}\hat{v}_{nt} &= \frac{\partial U(\bar{w}_{nt}, \bar{P}_{nt})}{\partial\ln(\bar{w}_{nt})}\mathbb{E}_{T-2}\hat{w}_{nt} + \frac{\partial U(\bar{w}_{nt}, \bar{P}_{nt})}{\partial\ln(\bar{P}_{nt})}\mathbb{E}_{T-2}\hat{P}_{nt} + \beta\sum_{i=1}^{N}\bar{\mu}_{nit}\mathbb{E}_{T-2}\hat{v}_{it+1} \\
\mathbb{E}_{T-2}\hat{\mu}_{nit} &= \frac{\beta}{\nu}\sum_{m=1}^{N}\Big(\mathbb{1}(m=i) - \bar{\mu}_{nmt}\Big)\mathbb{E}_{T-2}\hat{v}_{mt+1} \\
\mathbb{E}_{T-2}\hat{l}_{nt+1} &= \sum_{i=1}^{N}\frac{\bar{\mu}_{int}\bar{l}_{it}}{\bar{l}_{nt+1}}\Big(\mathbb{E}_{T-2}\hat{\mu}_{int} + \mathbb{E}_{T-2}\hat{l}_{it}\Big),
\end{aligned} \tag{A.16}$$

The approximation points are $\{\mathbb{E}_{T-1}\lambda_{nit}, \mathbb{E}_{T-1}P_{nt}, \mathbb{E}_{T-1}w_{nt}, \mathbb{E}_{T-1}l_{nt}, \mathbb{E}_{T-1}\mu_{nit}\}_{i=1,n=1,t=T-1}^{N,N,T}$. For example, $\bar{\lambda}_{int} \equiv \mathbb{E}_{T-1}\lambda_{nit}$ and $\bar{\mu}_{int} \equiv \mathbb{E}_{T-1}\mu_{nit}$.

Using the inputs described above and the fact that $\mathbb{E}_{T-2}\hat{l}_{iT-1}$=0, we can solve the system of the equations to obtain $\{\mathbb{E}_{T-2}\hat{\lambda}_{nit}, \mathbb{E}_{T-2}\hat{P}_{nt}, \mathbb{E}_{T-2}\hat{w}_{nt}, \mathbb{E}_{T-2}\hat{l}_{nt}, \mathbb{E}_{T-2}\hat{\mu}_{nit}, \mathbb{E}_{T-2}\hat{v}_t\}_{i=1,n=1,t=T-1}^{N,N,T}$. Note that the expectation operators here stand for the difference in the expected value of an outcome between expectations formed in $T-2$ and the expectation formed in $T-1$. For example, $\mathbb{E}_{T-2}\hat{v}_t \equiv \mathbb{E}_{T-2}v_t - \mathbb{E}_{T-1}\tilde{v}_t$ for $t \in \{T-1, T\}$. We then recover the level variables of these variables.

iv. Append the output from (iii) by $\{\tilde{\mu}_{niT-2}, \tilde{w}_{nT-2}, \tilde{P}_{iT-2}, \tilde{\lambda}_{iT-2}\}$, namely, agents' *actual* migration decision and the actual outcomes of the static trade equilibrium in period $T-2$. Together with the output from the previous step, this constitutes the solution to the agents' problem in period $T-2$.

v. Note that steps (iii) and (iv) take as input the beliefs of agents in period $T-1$ about outcomes in $t \in \{T-1, T\}$ and produce as output the belief of agents in period $T-2$ about outcomes in $t \in \{T-2, T-1, T\}$. Iterating on steps (iii) and (iv) backward recursively until the first period is reached delivers the belief paths in all periods.

With the belief paths at hand, we can approximate around each of these paths to obtain the perfect foresight path for agents in the corresponding periods. Both belief and perfect foresight paths can be used to recover counterfactual outcomes, as established in Proposition 1.

Extension for when the data end at $T' < T$. In the above discussion we have assumed that the researcher observes the actual allocation until the end of model T. Suppose the researcher only observed the outcome up until $T' < T$. We can proceed by first constructing the belief path of agents in period T' using a time-difference version of the model, which can be solved using the time-difference algorithm described in Caliendo et al. (2019) with first-order accuracy.[1] Doing so requires taking a stand on the period-by-period changes in the fundamentals between T' and T according to agents' belief in T', which is inevitable as no data are observed beyond T'.[2] With the belief path in T' at hand, iterating on steps (iii) and (iv) delivers the belief paths from all periods.

Measuring welfare using belief paths and the actual allocation. We now describe how to measure the welfare of the agents in this economy using the actual allocation and the outcomes on the belief paths. Let $v^t_{nt'} \equiv \mathbb{E}_t v_{nt'}$ be the expected value of location n in period t' under the belief from period $t \leq t'$. Let $\tilde{v}_{nt}$ be the actual value of location n at period t, which is defined as the ex-post value of a representative agent in location n. This value takes into account the fact that the people in each location make the best decision according to their beliefs and the realization of their idiosyncratic taste draws, but not the ex-post best decision given the realization of the fundamentals. Denote by T the last period of the model. Without loss of generality, let period $\tilde{T}$ be the first period agents obtain perfect foresight, so $v^{\tilde{T}}_{nt} = \tilde{v}_{nt}$ for $t \geq \tilde{T}$.[3] The actual location values at each moment in time $\tilde{v}_{nt}$ can be recovered as follows:

i. Compute $\tilde{v}^{\tilde{T}-1}_{n\tilde{T}}$ and the actual welfare $\tilde{v}_{nt}$ for $t \geq \tilde{T}$ using

$$\begin{aligned} \tilde{v}_{nt} &= \sum_{s=t}^{T} \beta^{s-t} U(\tilde{c}_{ns}) - \nu \sum_{s=t}^{T} \beta^{s-t} \ln \tilde{\mu}_{nns} \\ v^{\tilde{T}-1}_{n\tilde{T}} &= \sum_{s=\tilde{T}}^{T} \beta^{s-\tilde{T}} U(\mathbb{E}_{\tilde{T}-1} c_{ns}) - \nu \sum_{s=\tilde{T}}^{T} \beta^{s-\tilde{T}} \ln \mathbb{E}_{\tilde{T}-1} \mu_{nns}, \end{aligned} \tag{A.17}$$

where $\tilde{c}_{ns} \equiv \frac{\tilde{w}_{ns}}{\tilde{P}_{ns}}$ is the realized real wage at time s and $\mathbb{E}_{\tilde{T}-1} c_{ns}$ is the expected value according to belief at period $\tilde{T}-1$. The intuition behind both equations is that the value of a location is determined by the future consumption stream of that location, adjusted by the option value of moving to other locations, which is captured by the share of stayers. By evaluating U and the logarithm functions at the expected outcomes for agents' belief values, the second equation in (A.17) uses the certainty equivalence result.[4]

[1] If beliefs are deterministic, then the solution is exact.

[2] Intuitively, the observed data contain information on the level of fundamentals in T'. This information, coupled with the assumption on future changes in these fundamentals, enables the researcher to solve for the evolution according to agents' beliefs.

[3] If agents never have perfect foresight during the model horizon, set $\tilde{T} = T+1$ and start from the second step of the algorithm, imposing that the continuation value for decision at T is zero.

[4] This equation can be extended using the second-order approximation explained in Proposition 3, in which case the

ii. Using the above input, derive $\tilde{v}_{n\tilde{T}-1}$, the actual value of location n at $\tilde{T}-1$, recursively:

$$\begin{aligned}
\tilde{v}_{n\tilde{T}-1} =& U(\tilde{c}_{n\tilde{T}-1}) + \mathbb{E}_{\tilde{T}-1} \sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}} [\beta \tilde{v}_{i\tilde{T}} - m_{ni\tilde{T}-1} + \epsilon_i] \\
=& U(\tilde{c}_{n\tilde{T}-1}) + \mathbb{E}_{\tilde{T}-1} \sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}} [\beta v_{i\tilde{T}}^{\tilde{T}-1} - m_{ni\tilde{T}-1} + \epsilon_i] + \mathbb{E}_{\tilde{T}-1} \sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}} [\beta \tilde{v}_{i\tilde{T}} - \beta v_{i\tilde{T}}^{\tilde{T}-1}] \\
=& U(\tilde{c}_{n\tilde{T}-1}) + [\beta v_{n\tilde{T}-1}^{\tilde{T}} - \nu \ln(\tilde{\mu}_{nn\tilde{T}-1})] + \sum_{i=1}^{N} \tilde{\mu}_{ni\tilde{T}-1} [\beta \tilde{v}_{i\tilde{T}} - \beta v_{i\tilde{T}}^{\tilde{T}-1}],
\end{aligned} \tag{A.18}$$

where the first line uses the definition of $\tilde{v}_{n\tilde{T}-1}$ and the third line uses the property that the actual migration in period $\tilde{T}-1$ is made under the belief that the continuation value of location n is $v_{i\tilde{T}-1}^{\tilde{T}}$. Equation (A.18) essentially decomposes the actual value in location n into the value according to agents' belief (the first two terms) and a correction that reflects the difference between the belief and the realization (the third term). Plugging in $v_{n\tilde{T}}^{\tilde{T}-1}$ and $\tilde{v}_{i\tilde{T}}$ from step (i) delivers $\tilde{v}_{n\tilde{T}-1}$.

iii. Decompose the actual value in period $\tilde{T}-2$ analogously to equation (A.18)

$$\begin{aligned}
\tilde{v}_{n\tilde{T}-2} &= U(\tilde{c}_{n\tilde{T}-2}) + \mathbb{E}_{\tilde{T}-2} \sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}} [\beta v_{i\tilde{T}-1} - m_{ni\tilde{T}-2} + \epsilon_j] \\
&= U(\tilde{c}_{n\tilde{T}-2}) + \mathbb{E}_{\tilde{T}-2} \sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}} [\beta v_{i\tilde{T}-1}^{\tilde{T}-2} - m_{ni\tilde{T}-2} + \epsilon_i] + \mathbb{E}_{\tilde{T}-2} \sum_{i=1}^{N} \mathbb{1}_{\text{i is chosen}} [\beta \tilde{v}_{i\tilde{T}-1} - \beta v_{i\tilde{T}-1}^{\tilde{T}-2}] \\
&= U(\tilde{c}_{n\tilde{T}-2}) + [\beta v_{n\tilde{T}-1}^{\tilde{T}-2} - \nu \ln(\tilde{\mu}_{nn\tilde{T}-2})] + \sum_{i=1}^{N} \tilde{\mu}_{ni\tilde{T}-2} [\beta \tilde{v}_{i\tilde{T}-1} - \beta v_{i\tilde{T}-1}^{\tilde{T}-2}].
\end{aligned}$$

Note that $\tilde{v}_{i\tilde{T}-1}$ has been calculated from step (ii), and $v_{i\tilde{T}-1}^{\tilde{T}-2}$ can be calculated analogously to equation (A.17) using the outcomes on the path according to the belief formed in $\tilde{T}-2$.

iv. Apply step (iii) recursively backward to obtain $\tilde{v}_{n1}$. This is the value of location n for agents under the actual realization of fundamentals.

values computed will have second-order accuracy.

A.3 Proof of Proposition 3

A second-order approximation of equations (3)-(8) that characterizes the decisions and future expected outcomes for agents in period t (see Supplementary Appendix C.2 for derivations) is [5]

$$
\begin{aligned}
&\forall t' \geq t, \\
\mathbb{E}_t\hat{\mu}_{nit'} &= \sum_{m=1}^{N}\frac{\beta}{\nu}\Big(\mathbb{1}(m=i)-\bar{\mu}_{nmt'}\Big)\mathbb{E}_t\hat{v}_{mt'+1} \\
&+\frac{1}{2}\sum_{m,o}^{N}(\frac{\beta}{\nu})^2\cdot\Big[\mathbb{1}(m=i)\bar{\mu}_{nit'}(\mathbb{1}(o=i)-\bar{\mu}_{not'})-\bar{\mu}_{nmt'}\bar{\mu}_{nit'}[\mathbb{1}(o=i)-2\bar{\mu}_{not'}+\mathbb{1}(o=m)]\Big]\mathbb{E}_t\hat{v}_{mt'+1}\hat{v}_{ot'+1} \\
\mathbb{E}_t\hat{l}_{nt'+1} &= \sum_{i,m}^{N}\frac{\beta}{\nu}\bar{\psi}_{int'+1}\Big(\mathbb{1}(m=n)-\bar{\mu}_{imt'}\Big)\mathbb{E}_t\hat{v}_{mt'+1}+\sum_{m=1}^{N}\bar{\psi}_{mnt'+1}\mathbb{E}_t\hat{l}_{mt'} \\
&+\frac{1}{2}\sum_{m,o}^{N}(\frac{\beta}{\nu})^2\cdot\Big[-\sum_{i=1}^{N}\bar{\mu}_{imt'}\bar{\psi}_{int'+1}[\mathbb{1}(o=n)+\mathbb{1}(o=m)-2\bar{\mu}_{iot'}]+\mathbb{1}(m=n)\sum_{i}^{N}\bar{\psi}_{int'+1}(\mathbb{1}(o=n)-\bar{\mu}_{iot'}) \\
&-[\sum_{i=1}^{N}\bar{\psi}_{int'+1}(\mathbb{1}(m=n)-\bar{\mu}_{imt'})]\cdot[\sum_{i}^{N}\bar{\psi}_{int'+1}(\mathbb{1}(o=n)-\bar{\mu}_{iot'})]\Big]\mathbb{E}_t\hat{v}_{mt'+1}\hat{v}_{ot'+1} \\
&+\frac{1}{2}\sum_{m,o}^{N}[\mathbb{1}(m=o)\bar{\psi}_{ont'+1}-\bar{\psi}_{mnt'+1}\bar{\psi}_{ont'+1}]\,\mathbb{E}_t\hat{l}_{mt'}\hat{l}_{ot'} \\
&+\sum_{m,o}^{N}\left[\frac{\beta}{\nu}\bar{\psi}_{ont'+1}(\mathbb{1}(o=n)-\bar{\mu}_{omt'})-\frac{\beta}{\nu}\bar{\psi}_{ont'+1}[\sum_{i}^{N}\bar{\psi}_{int'+1}(\mathbb{1}(m=n)-\bar{\mu}_{imt'})]\right]\mathbb{E}_t\hat{v}_{mt'+1}\hat{l}_{ot'}, \\
\mathbb{E}_t\hat{\lambda}_{nit'} &= -\theta(\mathbb{E}_t\hat{w}_{it'}+\mathbb{E}_t\hat{\kappa}_{int'}-\mathbb{E}_t\hat{P}_{nt'})+\mathbb{E}_t\hat{z}_{it'} \qquad \text{(A.19)} \\
\mathbb{E}_t\hat{P}_{nt'} &= \sum_{i=1}^{N}\bar{\lambda}_{nit'}\mathbb{E}_t\hat{w}_{it'}+\sum_{i=1}^{N}\bar{\lambda}_{nit'}\cdot\mathbb{E}_t\hat{w}_{it'}+\sum_{i=1}^{N}\Big(-\frac{1}{\theta}\Big)\bar{\lambda}_{nit'}\mathbb{E}_t\hat{z}_{it'} \\
&+\frac{1}{2}\sum_{i,m}^{N}\theta\Big(-\mathbb{1}(i=m)\bar{\lambda}_{nit'}+\bar{\lambda}_{nit'}\bar{\lambda}_{nmt'}\Big)\cdot\Big(\mathbb{E}_t\hat{w}_{it'}\hat{w}_{mt'}+\mathbb{E}_t\hat{\kappa}_{nit'}\hat{\kappa}_{nmt'}+\Big(\frac{1}{\theta}\Big)^2\mathbb{E}_t\hat{z}_{it'}\hat{z}_{mt'}\Big) \\
&+\sum_{i,m}^{N}\theta\Big(-\mathbb{1}(i=m)\bar{\lambda}_{nit'}+\bar{\lambda}_{nit'}\bar{\lambda}_{nmt'}\Big)\Big(\mathbb{E}_t\hat{w}_{it'}\hat{\kappa}_{nmt'}-\Big(\frac{1}{\theta}\Big)\mathbb{E}_t\hat{w}_{it'}\hat{z}_{mt'}-\Big(\frac{1}{\theta}\Big)\mathbb{E}_t\hat{\kappa}_{nit'}\hat{z}_{mt'}\Big) \\
\mathbb{E}_t\hat{w}_{nt'}+\mathbb{E}_t\hat{l}_{nt'} &= \sum_{i=1}^{N}\frac{\bar{\lambda}_{int'}\bar{w}_{it'}\bar{l}_{it'}}{\bar{w}_{nt'}\bar{l}_{nt'}}\Big(\mathbb{E}_t\hat{\lambda}_{int'}+\mathbb{E}_t\hat{w}_{it'}+\mathbb{E}_t\hat{l}_{it'}\Big) \\
&+\frac{1}{2}\sum_{i,m}^{N}\Big(\mathbb{1}(m=i)\frac{\bar{\lambda}_{int'}\bar{w}_{it'}\bar{l}_{it}}{\bar{w}_{nt'}\bar{l}_{nt'}}-\frac{\bar{\lambda}_{int'}\bar{w}_{it'}\bar{l}_{it'}\bar{\lambda}_{mnt'}\bar{w}_{mt'}\bar{l}_{mt'}}{(\bar{w}_{nt'}\bar{l}_{nt'})^2}\Big)\Big(\mathbb{E}_t\hat{w}_{it'}\hat{w}_{mt'}+\mathbb{E}_t\hat{\lambda}_{int'}\hat{\lambda}_{mnt'}+\mathbb{E}_t\hat{l}_{it'}\hat{l}_{mt'}\Big) \\
&+\sum_{i,m}^{N}\Big(\mathbb{1}(m=i)\frac{\bar{\lambda}_{int'}\bar{w}_{it'}\bar{l}_{it'}}{\bar{w}_{nt'}\bar{l}_{nt'}}-\frac{\bar{\lambda}_{int'}\bar{w}_{it'}\bar{l}_{it'}\bar{\lambda}_{mnt'}\bar{w}_{mt'}\bar{l}_{mt'}}{(\bar{w}_{nt'}\bar{l}_{nt'})^2}\Big)\Big(\mathbb{E}_t\hat{\lambda}_{int'}\hat{w}_{mt'}+\mathbb{E}_t\hat{\lambda}_{int'}\hat{l}_{mt'}+\mathbb{E}_t\hat{w}_{it'}\hat{l}_{mt'}\Big) \\
\mathbb{E}_t\hat{v}_{nt'+1} &= \frac{\partial U(\bar{w}_{nt'+1},\bar{P}_{nt'+1})}{\partial\ln(\bar{w}_{nt'+1})}\mathbb{E}_t\hat{w}_{nt'+1}+\frac{\partial U(\bar{w}_{nt'+1},\bar{P}_{nt'+1})}{\partial\ln(\bar{P}_{nt'+1})}\mathbb{E}_t\hat{P}_{nt'+1} \\
&+\frac{1}{2}\frac{\partial^2U(\bar{w}_{nt'+1},\bar{P}_{nt'+1})}{\partial\ln(\bar{w}_{nt'+1})^2}\mathbb{E}_t\hat{w}^2_{nt'+1}+\frac{1}{2}\frac{\partial^2U(\bar{w}_{nt'+1},\bar{P}_{nt'+1})}{\partial\ln(\bar{P}_{nt'+1})^2}\mathbb{E}_t\hat{P}^2_{nt'+1}+\frac{\partial^2U(\bar{w}_{nt'+1},\bar{P}_{nt'+1})}{\partial\ln(\bar{w}_{nt'+1})\partial\ln(\bar{P}_{nt'+1})}\mathbb{E}_t\hat{w}_{nt'+1}\hat{P}_{nt'+1} \\
&+\sum_{m=1}^{N}\beta\bar{\mu}_{nmt'+1}\mathbb{E}_t\hat{v}_{mt'+2}+\frac{1}{2}\sum_{o,m}^{N}\frac{\beta^2}{\nu}\bar{\mu}_{not'+1}\Big(\mathbb{1}(m=o)-\bar{\mu}_{nmt'+1}\Big)\mathbb{E}_t\hat{v}_{ot'+2}\hat{v}_{mt'+2},
\end{aligned}
$$

where $\bar{\psi}_{int'+1}=\frac{\bar{\mu}_{int'}\bar{l}_{it'}}{\bar{l}_{nt'+1}}$. The equilibrium objects are $\Big\{\mathbb{E}_t\hat{v}_{it'+1},\mathbb{E}_t\hat{\mu}_{nit'},\mathbb{E}_t\hat{l}_{nt'},\mathbb{E}_t\hat{w}_{it'},\mathbb{E}_t\hat{P}_{it'}\Big\}_{i=1,n=1,t'=t}^{N,N,T}$ and the additional inputs compared to Proposition 1 are the covariance of these variables, e.g., $\big\{\mathbb{E}_t\hat{v}_{it'}\hat{v}_{nt'},\mathbb{E}_t\hat{w}_{it'}\hat{w}_{nt'},\mathbb{E}_t\hat{P}_{it'}\hat{P}_{nt'}\big\}$.

We develop the following simulation-based algorithm, which is second-order accurate.

i. Start from period $t=t_0$, simulate S paths of $z_{t_0}^T\equiv(z_{t_0},z_{t_0+1},z_{t_0+2},...z_T)$.

ii. Solve the first-order approximation for each $s=1,..,S$ using Proposition 1. Denote the solution $\hat{x}_{t'}(s)$ for all $t'\geq t_0$, where $\hat{x}\in\{\hat{w}_{nt'},\hat{P}_{nt'},\hat{v}_{t'+1},etc.\}$.

[5]When t'=t, the expectation operators in the period-t outcomes are redundant. For example, $\mathbb{E}_t\hat{P}_{nt}=\hat{P}_{nt}$.

iii. Approximate $\mathbb{E}_{t_0}\hat{x}_{t'}\hat{y}_{t'} \approx \frac{1}{S}\sum_{s=1}^{S}\hat{x}_{t'}(s)\hat{y}_{t'}(s)$ and then plug $\mathbb{E}_{t_0}\hat{x}_{t'}\hat{y}_{t'}$ into the system of equations (A.19) and solve for the remaining first-order terms, which gives the decision in period t_0 and the expected outcomes for $t' > t_0$ based on the belief in t_0.

iv. Move the economy to $t = t_0 + 1$, repeat the above process until $t = T$.

A.4 Proof of Proposition 4

In this section, we first illustrate the curse of dimensionality in computing the equilibrium under heterogeneous beliefs with full rationality. We then show that the problem is tractable under each of the three assumptions on higher-order beliefs. In each case, we briefly discuss the numerical algorithm.

A.4.1 Heterogenous Beliefs with Full Rationality

Without loss of generality, we denote the two groups of agents by A and B, with their beliefs given by $f^g(z_{t+1}|z^t)$, $g \in \{A, B\}$. Let $\mathbb{E}_t^g z_{t+1}$ be the expected productivity for period $t+1$ under type $g \in \{A, B\}$'s belief at t, that is, $\mathbb{E}_t^g z_{t+1} = \int_{z_{t+1}} z_{t+1} f^g(z_{t+1}|z^t)dz_{t+1}$. Then, the expected value of location n for type g agent is

$$\begin{aligned}
\mathbb{E}^g[v_{nt+1}^g(z^{t+1})|z^t] &= \int_{z_{t+1}} U(c_{nt}^g(z^t)) + \nu \ln\left(\sum_{i=1}^{N}\exp\left(\beta\mathbb{E}_t^g[v_{it+2}^g(z^{t+1})|z^t] - m_{nit}\right)^{1/\nu}\right) f^g(z_{t+1}|z^t)dz_{t+1} \\
&\equiv \mathbb{E}_t^g v_{nt+1}^g
\end{aligned} \tag{A.20}$$

Similarly, the migration decision for type g is given by $\mu_{nit}^g(z^t) = \frac{\exp\left(\beta\mathbb{E}_t^g v_{it+1}^g - m_{nit}\right)^{1/\nu}}{\sum_{h=1}^{N}\exp\left(\beta\mathbb{E}_t^g v_{ht+1}^g - m_{nht}\right)^{1/\nu}}$.

While migration decisions take the same form as they do in equation (3), wage $w_{nt}(z^t)$ and price $P_{nt}(z^t)$ are determined by the decisions of both types, which are given by the solution to the following system of equations:

$$\begin{aligned}
\lambda_{nit}(z^t) &= z_{it}\left(\frac{w_{it}(z^t)\kappa_{nit}}{P_{nt}(z^t)}\right)^{-\theta} \\
P_{nt}(z^t) &= \left[\sum_{i=1}^{N} z_{it}\left(w_{it}(z^t)\kappa_{nit}\right)^{-\theta}\right]^{-1/\theta} \\
w_{nt}(z^t)l_{nt}(z^t) &= \sum_{i=1}^{N}\lambda_{int}(z^t)w_{it}(z^t)l_{it}(z^t) \\
l_{nt+1}(z^t) &= \sum_{g} l_{nt+1}^g(z^t) \\
l_{nt+1}^g(z^t) &= \sum_{m=1}^{N}\mu_{mnt}^g(z^t)l_{mt}^g(z^{t-1}), \text{ for } g \in \{A, B\}
\end{aligned} \tag{A.21}$$

Decision of A. We first consider the decision of type-A agents in period t. Following the approach of Proposition 1, we approximate equations for each z^t around a perfect foresight path and integrate across all z^t for each t using $f^A(z_{t+1}|z^t)$. For example, type A agents' expectation on future productivity given history z^t is defined by

$$\begin{aligned}
\mathbb{E}_t^A z_{t+1} &\equiv \int_{z_{t+1}} z_{t+1} f^A(z_{t+1}|z^t) dz_{t+1} \\
\mathbb{E}_t^A z_{t+2} &\equiv \int_{z_{t+2}} z_{t+2} f^A(z_{t+2}|z^t) dz_{t+2} \\
&= \int_{z_{t+2}} z_{t+2} f^A(z_{t+2}|z^{t+1}) f^A(z_{t+1}|z^t) dz_{t+2}. \qquad \text{(A.22)}
\end{aligned}$$

Doing so delivers the following system of linear expectation equations that characterizes the decision of type A at period t, taking as given A's expectation about B's future decisions.

For period t,

$$\begin{aligned}
\mathbb{E}_t^A \hat{v}_{nt+1}^A &= \frac{\partial U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{w}_{nt+1})} \mathbb{E}_t^A \hat{w}_{nt+1} + \frac{\partial U(\bar{w}_{nt+1}, \bar{P}_{nt+1})}{\partial \ln(\bar{P}_{nt+1})} \mathbb{E}_t^A \hat{P}_{nt+1} \\
&\quad + \beta \sum_{i=1}^{N} \bar{\mu}_{nit+1} \mathbb{E}_t^A \hat{v}_{nt+2}^A, \qquad \text{(A.23)} \\
\hat{\mu}_{nit}^A &= \frac{\beta}{\nu} \mathbb{E}_t^A \hat{v}_{it+1}^A - \frac{\beta}{\nu} \sum_{m=1}^{N} \bar{\mu}_{nmt} \mathbb{E}_t^A \hat{v}_{mt+1}^A, \qquad \text{(A.24)} \\
\hat{l}_{nt+1}^A &= \sum_{i=1}^{N} \bar{\psi}_{int+1}^A \left(\hat{\mu}_{int}^A + \hat{l}_{it}^A \right), \qquad \text{(A.25)} \\
\hat{l}_{nt+1}^B &= \sum_{i=1}^{N} \bar{\psi}_{int+1}^B \left(\hat{\mu}_{int}^B + \hat{l}_{it}^B \right), \qquad \text{(A.26)} \\
\hat{l}_{nt+1} &= \frac{\bar{l}_{nt+1}^A}{\bar{l}_{nt+1}} \hat{l}_{nt+1}^A + \frac{\bar{l}_{nt+1}^B}{\bar{l}_{nt+1}} \hat{l}_{nt+1}^B, \qquad \text{(A.27)} \\
\mathbb{E}_t^A \hat{\lambda}_{nit+1} &= -\theta \left(\mathbb{E}_t^A \hat{w}_{it+1} - \mathbb{E}_t^A \hat{P}_{nt+1} + \mathbb{E}_t^A \hat{\kappa}_{nit+1} \right) + \mathbb{E}_t^A \hat{z}_{it+1}, \qquad \text{(A.28)} \\
\mathbb{E}_t^A \hat{P}_{nt+1} &= \sum_{i=1}^{N} \bar{\lambda}_{nit+1} \Big(\mathbb{E}_t^A \hat{w}_{it+1} + \mathbb{E}_t^A \hat{\kappa}_{nit+1} - \frac{1}{\theta} \mathbb{E}_t^A \hat{z}_{it+1} \Big), \qquad \text{(A.29)} \\
\mathbb{E}_t^A \hat{w}_{nt+1} + \mathbb{E}_t^A \hat{l}_{nt+1} &= \sum_{i=1}^{N} \frac{\bar{\lambda}_{int+1} \bar{w}_{nt+1} \bar{l}_{nt+1}}{\bar{w}_{it+1} \bar{l}_{it+1}} \Big(\mathbb{E}_t^A \hat{\lambda}_{int+1} + \mathbb{E}_t^A \hat{w}_{it+1} + \mathbb{E}_t^A \hat{l}_{it+1} \Big) \qquad \text{(A.30)}
\end{aligned}$$

For periods $t' > t$,

$$\begin{aligned}
\mathbb{E}_t^A \hat{v}_{nt'+1}^A &= \frac{\partial U(\bar{w}_{nt'+1}, \bar{P}_{nt'+1})}{\partial \ln(\bar{w}_{nt'+1})} \mathbb{E}_t^A \hat{w}_{nt'+1} + \frac{\partial U(\bar{w}_{nt'+1}, \bar{P}_{nt'+1})}{\partial \ln(\bar{P}_{nt'+1})} \mathbb{E}_t^A \hat{P}_{nt'+1} \\
&\quad + \beta \sum_{i=1}^{N} \bar{\mu}_{nit'+1} \mathbb{E}_t^A \hat{v}_{nt'+2}^A, && \text{(A.31)} \\
\mathbb{E}_t^A \hat{\mu}_{nit'}^A &= \frac{\beta}{\nu} \mathbb{E}_t^A \hat{v}_{it'+1}^A - \frac{\beta}{\nu} \sum_{m=1}^{N} \bar{\mu}_{nmt'} \mathbb{E}_t^A \hat{v}_{mt'+1}^A, && \text{(A.32)} \\
\mathbb{E}_t^A \hat{l}_{nt'+1}^A &= \sum_{i=1}^{N} \bar{\psi}_{int'+1}^A \left(\mathbb{E}_t^A \hat{\mu}_{int'}^A + \mathbb{E}_t^A \hat{l}_{it'}^A \right), && \text{(A.33)} \\
\mathbb{E}_t^A \hat{l}_{nt'+1}^B &= \sum_{i=1}^{N} \bar{\psi}_{int'+1}^B \left(\mathbb{E}_t^A \hat{\mu}_{int'}^B + \mathbb{E}_t^A \hat{l}_{it'}^B \right), && \text{(A.34)} \\
\mathbb{E}_t^A \hat{l}_{nt'+1} &= \frac{\bar{l}_{nt'+1}^A}{\bar{l}_{nt'+1}} \mathbb{E}_t^A \hat{l}_{nt'+1}^A + \frac{\bar{l}_{nt'+1}^B}{\bar{l}_{nt'+1}} \mathbb{E}_t^A \hat{l}_{nt'+1}^B, && \text{(A.35)} \\
\mathbb{E}_t^A \hat{\lambda}_{nit'} &= -\theta \left(\mathbb{E}_t^A \hat{w}_{it'} - \mathbb{E}_t^A \hat{P}_{nt'} + \mathbb{E}_t^A \hat{\kappa}_{nit'} \right) + \mathbb{E}_t^A \hat{z}_{it'}, && \text{(A.36)} \\
\mathbb{E}_t^A \hat{P}_{nt'} &= \sum_{i=1}^{N} \bar{\lambda}_{nit'} \left(\mathbb{E}_t^A \hat{w}_{it'} + \mathbb{E}_t^A \hat{\kappa}_{nit'} - \frac{1}{\theta} \mathbb{E}_t^A \hat{z}_{it'} \right), && \text{(A.37)} \\
\mathbb{E}_t^A \hat{w}_{nt'} + \mathbb{E}_t^A \hat{l}_{nt'} &= \sum_{i=1}^{N} \frac{\bar{\lambda}_{int'} \bar{w}_{nt'} \bar{l}_{nt'}}{\bar{w}_{it'} \bar{l}_{it'}} \left(\mathbb{E}_t^A \hat{\lambda}_{int'} + \mathbb{E}_t^A \hat{w}_{it'} + \mathbb{E}_t^A \hat{l}_{it'} \right) && \text{(A.38)}
\end{aligned}$$

where $\bar{\psi}_{int+1}^g \equiv \frac{\bar{\mu}_{int}^g \bar{l}_{it}^g}{\bar{l}_{nt'+1}^g}$ for $g \in \{A, B\}$ and $\bar{l}_{nt+1} = \bar{l}_{nt+1}^A + \bar{l}_{nt+1}^B$. These equations together constitute a solution to agent A's period t problem.

As equations (A.26) and (A.34) make clear, to solve for A's decision in period t, we need not only B's period-t migration decision $\hat{\mu}_{nit}^B$ but also A's belief about B's decisions $\mathbb{E}_t^A \mu_{nit'}^B$ for all $t' > t$. Such beliefs in turn depend on B's own belief in t' about the future values for $t' > t$:

$$\mathbb{E}_t^A \hat{\mu}_{nit'}^B = \frac{\beta}{\nu} \mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{v}_{it'+1}^B - \frac{\beta}{\nu} \sum_{m=1}^{N} \bar{\mu}_{nmt'} \beta \mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{v}_{mt'+1}^B, \quad \text{(A.39)}$$

where $\mathbb{E}_{t'}^B$ is defined by B's beliefs $f^B(z_{t'+1}|z^{t'})$. The system of equations that characterizes this is

For periods $t'' \geq t'$**,**

$$\begin{aligned}
\mathbb{E}^B_{t'}\hat{v}^B_{nt''+1} &= \frac{\partial U(\bar{w}_{nt''+1},\bar{P}_{nt''+1})}{\partial \ln(\bar{w}_{nt''+1})}\mathbb{E}^B_{t'}\hat{w}_{nt''+1} + \frac{\partial U(\bar{w}_{nt''+1},\bar{P}_{nt''+1})}{\partial \ln(\bar{P}_{nt''+1})}\mathbb{E}^B_{t'}\hat{P}_{nt''+1} \\
&\quad +\beta\sum_{i=1}^{N}\bar{\mu}_{nit''+1}\mathbb{E}^B_{t'}\hat{v}^B_{nt''+2}, && \text{(A.40)} \\
\mathbb{E}^B_{t'}\hat{\mu}^B_{nit''} &= \frac{\beta}{\nu}\mathbb{E}^B_{t'}\hat{v}^B_{it''+1} - \frac{\beta}{\nu}\sum_{m=1}^{N}\bar{\mu}_{nmt''}\beta\mathbb{E}^B_{t'}\hat{v}^B_{mt''+1}, && \text{(A.41)} \\
\mathbb{E}^B_{t'}\hat{l}^A_{nt''+1} &= \sum_{i=1}^{N}\bar{\psi}^A_{int''+1}\left(\mathbb{E}^B_{t'}\hat{\mu}^A_{int''} + \mathbb{E}^B_{t'}\hat{l}^A_{it''}\right), && \text{(A.42)} \\
\mathbb{E}^B_{t'}\hat{l}^B_{nt''+1} &= \sum_{i=1}^{N}\bar{\psi}^B_{int''+1}\left(\mathbb{E}^B_{t'}\hat{\mu}^B_{int''} + \mathbb{E}^B_{t'}\hat{l}^B_{it''}\right), && \text{(A.43)} \\
\mathbb{E}^B_{t'}\hat{l}_{nt''+1} &= \frac{\bar{l}^A_{nt''+1}}{\bar{l}_{nt''+1}}\mathbb{E}^B_{t'}\hat{l}^A_{nt''+1} + \frac{\bar{l}^B_{nt''+1}}{\bar{l}_{nt''+1}}\mathbb{E}^B_{t'}\hat{l}^B_{nt''+1}, && \text{(A.44)} \\
\mathbb{E}^B_{t'}\hat{\lambda}_{nit''} &= -\theta\left(\mathbb{E}^B_{t'}\hat{w}_{it'} - \mathbb{E}^B_{t'}\hat{P}_{nt''} + \mathbb{E}^B_{t'}\hat{\kappa}_{nit''}\right) + \mathbb{E}^B_{t'}\hat{z}_{it''}, && \text{(A.45)} \\
\mathbb{E}^B_{t'}\hat{P}_{nt''} &= \sum_{i=1}^{N}\bar{\lambda}_{nit''}\left(\mathbb{E}^B_{t'}\hat{w}_{it''} + \mathbb{E}^B_{t'}\hat{\kappa}_{nit''} - \frac{1}{\theta}\mathbb{E}^B_{t'}\hat{z}_{it''}\right), && \text{(A.46)} \\
\mathbb{E}^B_{t'}\hat{w}_{nt''} + \mathbb{E}^B_{t'}\hat{l}_{nt''} &= \sum_{i=1}^{N}\frac{\bar{\lambda}_{int''}\bar{w}_{nt''}\bar{l}_{nt''}}{\bar{w}_{it''}\bar{l}_{it''}}\left(\mathbb{E}^B_{t'}\hat{\lambda}_{int''} + \mathbb{E}^B_{t'}\hat{w}_{it''} + \mathbb{E}^B_{t'}\hat{l}_{it''}\right). && \text{(A.47)}
\end{aligned}$$

Note that to solve for $\hat{\mu}^B_{nit'}$, this system of equations needs be solved for all $t'' = t', ..., T$.

Therefore, *in the eyes of type A agents at period t*, type B's migration decision at t' is characterized by

For periods $t'' \geq t'$**,**

$$\begin{aligned}
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^B_{nt''+1} &= \frac{\partial U(\bar{w}_{nt''+1},\bar{P}_{nt''+1})}{\partial \ln(\bar{w}_{nt''+1})}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{nt''+1} + \frac{\partial U(\bar{w}_{nt''+1},\bar{P}_{nt''+1})}{\partial \ln(\bar{P}_{nt''+1})}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{P}_{nt''+1} \\
&\quad +\beta\sum_{i=1}^{N}\bar{\mu}_{nit''+1}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^B_{nt''+2}, && \text{(A.48)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\mu}^B_{nit''} &= \frac{\beta}{\nu}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^B_{it'+1} - \frac{\beta}{\nu}\sum_{m=1}^{N}\bar{\mu}_{nmt''}\beta\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^B_{mt'+1}, && \text{(A.49)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^A_{nt''+1} &= \sum_{i=1}^{N}\bar{\psi}^A_{int''+1}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\mu}^A_{int''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^A_{it'}\right), && \text{(A.50)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^B_{nt''+1} &= \sum_{i=1}^{N}\bar{\psi}^B_{int''+1}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\mu}^B_{int''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^B_{it'}\right), && \text{(A.51)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}_{nt''+1} &= \frac{\bar{l}^A_{nt''+1}}{\bar{l}_{nt''+1}}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^A_{nt''+1} + \frac{\bar{l}^B_{nt''+1}}{\bar{l}_{nt''+1}}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^B_{nt''+1}, && \text{(A.52)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\lambda}_{nit''} &= -\theta\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{it''} - \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{P}_{nt''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\kappa}_{nit''}\right) + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{z}_{it''}, && \text{(A.53)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{P}_{nt''} &= \sum_{i=1}^{N}\bar{\lambda}_{nit''}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{it''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\kappa}_{nit''} - \frac{1}{\theta}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{z}_{it''}\right), && \text{(A.54)} \\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{nt''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}_{nt''} &= \sum_{i=1}^{N}\frac{\bar{\lambda}_{int''}\bar{w}_{nt''}\bar{l}_{nt''}}{\bar{w}_{it''}\bar{l}_{it''}}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\lambda}_{int''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{it''} + \mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}_{it''}\right). && \text{(A.55)}
\end{aligned}$$

Thus, to solve for A's decision in period t, we need to solve the above system of equations for all

$t' \geq t$ and for all $t'' \geq t'$. Further notice that A's belief on B's belief about A's future migration decision, $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int''}^A$, appears in equation (A.50). Computing this object requires solving an additional system of equations that is analogous to the equations that define $\hat{\mu}_{int''}^A$ ((A.31) to (A.38)), with the difference being the $\mathbb{E}_t^A \mathbb{E}_{t'}^B$ operator now appears before all endogenous outcomes and exogenous shocks. This means we will need to solve for third-order beliefs on all outcomes. As B's migration decision also appears in this system of equations, this process continues. The implication of this process is that the higher-order beliefs about future fundamentals and endogenous variables, e.g., $\mathbb{E}_1^A \hat{z}_t$, $\mathbb{E}_1^A \mathbb{E}_2^B \hat{z}_t$, $\mathbb{E}_1^A \mathbb{E}_2^B \mathbb{E}_3^A \hat{z}_t$, $\mathbb{E}_1^A \mathbb{E}_2^B \mathbb{E}_3^A \mathbb{E}_4^B \hat{z}_t$, ... are relevant to the decision of agent A in the first period. Solving this problem, therefore, requires calculating all higher-order beliefs about fundamentals and solving the higher-order beliefs about agents' endogenous decisions.

We now describe how our assumptions on higher-order beliefs retain tractability.

A.4.2 Assumption 1

An immediate consequence of Assumption 1 is that, by the Law of Iterative Expectations, higher-order beliefs about the fundamentals of each type $g', g \in \{A, B\}, g \neq g'$ can be simplified as

$$\mathbb{E}_t^g \mathbb{E}_{t'}^{g'} \hat{z}_{t''} = \mathbb{E}_t^g \hat{z}_{t''} \text{ for } t' \geq t+1 \text{ and } t'' \geq t'. \tag{A.56}$$

This implies we can simplify the expectation operators in (A.48) to (A.55) to $\mathbb{E}_t^A$. This delivers

For periods $t'' > t$,

$$\begin{aligned}
\mathbb{E}_t^A \hat{v}_{nt''+1}^B &= \frac{\partial U(\bar{w}_{nt''+1}, \bar{P}_{nt''+1})}{\partial \ln(\bar{w}_{nt''+1})} \mathbb{E}_t^A \hat{w}_{nt''+1} + \frac{\partial U(\bar{w}_{nt''+1}, \bar{P}_{nt''+1})}{\partial \ln(\bar{P}_{nt''+1})} \mathbb{E}_t^A \hat{P}_{nt''+1} \\
&\quad + \beta \sum_{i=1}^N \bar{\mu}_{nit''+1} \mathbb{E}_t^A \hat{v}_{nt''+2}^B, && \text{(A.57)} \\
\mathbb{E}_t^A \hat{\mu}_{nit''}^B &= \frac{\beta}{\nu} \mathbb{E}_t^A \hat{v}_{it''+1}^B - \frac{\beta}{\nu} \sum_{m=1}^N \bar{\mu}_{nmt''} \beta \mathbb{E}_t^A \hat{v}_{mt''+1}^B, && \text{(A.58)} \\
\mathbb{E}_t^A \hat{l}_{nt''+1}^A &= \sum_{i=1}^N \bar{\psi}_{int''+1}^A \left(\mathbb{E}_t^A \hat{\mu}_{int''}^A + \mathbb{E}_t^A \hat{l}_{it''}^A \right), && \text{(A.59)} \\
\mathbb{E}_t^A \hat{l}_{nt''+1}^B &= \sum_{i=1}^N \bar{\psi}_{int''+1}^B \left(\mathbb{E}_t^A \hat{\mu}_{int''}^B + \mathbb{E}_t^A \hat{l}_{it''}^B \right), && \text{(A.60)} \\
\mathbb{E}_t^A \hat{l}_{nt''+1} &= \frac{\bar{l}_{nt''+1}^A}{\bar{l}_{nt''+1}} \mathbb{E}_t^A \hat{l}_{nt''+1}^A + \frac{\bar{l}_{nt''+1}^B}{\bar{l}_{nt''+1}} \mathbb{E}_t^A \hat{l}_{nt''+1}^B, && \text{(A.61)} \\
\mathbb{E}_t^A \hat{\lambda}_{nit''} &= -\theta \left(\mathbb{E}_t^A \hat{w}_{it''} - \mathbb{E}_t^A \hat{P}_{nt''} + \mathbb{E}_t^A \hat{\kappa}_{nit''} \right) + \mathbb{E}_t^A \hat{z}_{it''}, && \text{(A.62)} \\
\mathbb{E}_t^A \hat{P}_{nt''} &= \sum_{i=1}^N \bar{\lambda}_{nit''} \left(\mathbb{E}_t^A \hat{w}_{it''} + \mathbb{E}_t^A \hat{\kappa}_{nit''} - \frac{1}{\theta} \mathbb{E}_t^A \hat{z}_{it''} \right), && \text{(A.63)} \\
\mathbb{E}_t^A \hat{w}_{nt''} + \mathbb{E}_t^A \hat{l}_{nt''} &= \sum_{i=1}^N \frac{\bar{\lambda}_{int''} \bar{w}_{nt''} \bar{l}_{nt''}}{\bar{w}_{it''} \bar{l}_{it''}} \left(\mathbb{E}_t^A \hat{\lambda}_{int''} + \mathbb{E}_t^A \hat{w}_{it''} + \mathbb{E}_t^A \hat{l}_{it''} \right). && \text{(A.64)}
\end{aligned}$$

Note that equations (A.31)-(A.38) and (A.57)-(A.64) take the same form. Intuitively, as group A believes that in the next period, group B will be convinced of the same belief, in *the eyes of group A*, the future value and migration decision for group B will be made under the same belief as that for group A.

Therefore, we can replace $E_t^A \hat{\mu}_{t'}^B$ by $E_t^A \hat{\mu}_{t'}^A$ in (A.34) and solve the system of linear equations for group A, i.e., equations (A.23) to (A.38), analogously to Proposition 1, taking type B's *current* period migration decision $\hat{\mu}^B_{int}$ as given.[6]

The above discussion characterizes $\hat{\mu}^A_{int}$, taking as given $\hat{\mu}^B_{int}$. We can proceed analogously and characterize $\hat{\mu}^B_{int}$, taking as given $\hat{\mu}^A_{int}$. For the solution to be an equilibrium, $\hat{\mu}^A_{int}$ and $\hat{\mu}^B_{int}$ need to be consistent with each other.

We solve for the decision in period t and the belief of each type in period t about the future outcomes using the following algorithm:

i. Given a guess for $\hat{\mu}^B_{nit}$, solve equations (A.23) to (A.38) for A's decision $\hat{\mu}^A_{nit}$ and expectation for future outcomes.

ii. Given a guess for $\hat{\mu}^A_{nit}$, solve equations analogous to (A.23) to (A.38) for B's decision $\hat{\mu}^B_{nit}$ and expectation for future outcomes.

iii. Iterate on (i) and (ii) until convergence, which delivers $\hat{\mu}^A_{nit}$ and $\hat{\mu}^B_{nit}$.

These steps move the economy from period t to $t+1$. Iterating on these steps for t, t+1, t+2,... delivers the evolution of the economy under heterogeneous beliefs.

A.4.3 Assumption 2

Assumption 2 implies that the third- and higher-order beliefs of type $g', g \in \{A, B\}, g \neq g'$ can be reduced to second-order beliefs. For example,

$$\mathbb{E}^g_t \mathbb{E}^{g'}_{t'} \mathbb{E}^g_{t''} \hat{z}_{t'''} = \mathbb{E}^g_t \mathbb{E}^{g'}_{t'} \hat{z}_{t'''} \neq \mathbb{E}^g_t \hat{z}_{t'''} \text{ for } t' \geq t+1, t'' \geq t'+1, \text{ and } t''' \geq t''. \tag{A.65}$$

Decision of A. We first consider type A's decision in period t. Following the discussion at the beginning of this section, Type A's decision at t is given by equations (A.23)-(A.38).

Entering this problem is $\mathbb{E}^A_t \hat{\mu}^B_{int'}$ for all future periods $t' > t$. For each t', $\mathbb{E}^A_t \hat{\mu}^B_{int'}$ is characterized by equations (A.48)-(A.55) that must hold for all $t'' \geq t'$. In turn, entering these equations is $\mathbb{E}^A_t \mathbb{E}^B_{t'} \hat{\mu}^A_{int''}$, which depends on A's beliefs in t'' about future values in $t''' \geq t''$ via

$$\mathbb{E}^A_t \mathbb{E}^B_{t'} \mathbb{E}^A_{t''} \hat{\mu}^A_{nit'''} = \frac{\beta}{\nu} \mathbb{E}^A_t \mathbb{E}^B_{t'} \mathbb{E}^A_{t''} \hat{v}^A_{it'''+1} - \frac{\beta}{\nu} \sum_{m=1}^{N} \bar{\mu}_{nmt''} \beta \mathbb{E}^A_t \mathbb{E}^B_{t'} \mathbb{E}^A_{t''} \hat{v}^A_{mt'''+1}. \tag{A.66}$$

Therefore, type A's belief at t on type B's belief at t' on type A's decision from t''' is characterized by the following system of equations:

[6] Group B's *current* migration decisions cannot be replaced because under Assumption 1, A is aware B has a different current belief about the future. The assumption only restricts A's beliefs about B's future beliefs.

In all periods $t''' \geq t''$, for any $t'' \geq t'$, and any $t' \geq t$,

$$\begin{aligned}
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{v}^A_{nt'''+1} &= \frac{\partial U(\bar{w}_{nt'''+1},\bar{P}_{nt'''+1})}{\partial\ln(\bar{w}_{nt'''+1})}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{w}_{nt'''+1} \\
&+\frac{\partial U(\bar{w}_{nt'''+1},\bar{P}_{nt'''+1})}{\partial\ln(\bar{P}_{nt'''+1})}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{P}_{nt'''+1}+\beta\sum_{i=1}^N\bar{\mu}_{nit'''+1}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{v}^A_{nt'''+2}, \qquad (A.67)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\mu}^A_{nit'''} &= \frac{\beta}{\nu}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{v}^A_{it'''+1}-\frac{\beta}{\nu}\sum_{m=1}^N\bar{\mu}_{nmt''}\beta\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{v}^A_{mt'''+1}, \qquad (A.68)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}^A_{nt''+1} &= \sum_{i=1}^N\bar{\psi}^A_{int'''+1}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\mu}^A_{int'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}^A_{it'''}\right), \qquad (A.69)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}^B_{nt'''+1} &= \sum_{i=1}^N\bar{\psi}^B_{int'''+1}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\mu}^B_{int'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}^B_{it'''}\right), \qquad (A.70)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}_{nt'''+1} &= \frac{\bar{l}^A_{nt'''+1}}{\bar{l}_{nt'''+1}}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}^A_{nt'''+1}+\frac{\bar{l}^B_{nt'''+1}}{\bar{l}_{nt'''+1}}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}^B_{nt'''+1}, \qquad (A.71)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\lambda}_{nit'''} &= -\theta\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{w}_{it'''}-\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{P}_{nt'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\kappa}_{nit'''}\right)\\
&+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{z}_{it'''}, \qquad (A.72)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{P}_{nt'''} &= \sum_{i=1}^N\bar{\lambda}_{nit'''}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{w}_{it'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\kappa}_{nit'''}-\frac{1}{\theta}\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{z}_{it'''}\right), \qquad (A.73)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{w}_{nt'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}_{nt'''} &= \sum_{i=1}^N\frac{\bar{\lambda}_{int'''}\bar{w}_{nt'''}\bar{l}_{nt'''}}{\bar{w}_{it'''}\bar{l}_{it'''}}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{\lambda}_{int'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{w}_{it'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}\hat{l}_{it'''}\right). \qquad (A.74)
\end{aligned}$$

This system of equations shows how third-order beliefs enter the problem. Under Assumption 2, $\mathbb{E}^A_t\mathbb{E}^B_{t'}\mathbb{E}^A_{t''}$ is reduced to $\mathbb{E}^A_t\mathbb{E}^B_{t'}$ such that the system of equations simplifies to

In all periods $t''' \geq t'$, for any $t' \geq t$,

$$\begin{aligned}
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^A_{nt'''+1} &= \frac{\partial U(\bar{w}_{nt'''+1},\bar{P}_{nt'''+1})}{\partial\ln(\bar{w}_{nt'''+1})}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{nt'''+1}+\frac{\partial U(\bar{w}_{nt'''+1},\bar{P}_{nt'''+1})}{\partial\ln(\bar{P}_{nt'''+1})}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{P}_{nt'''+1}\\
&+\beta\sum_{i=1}^N\bar{\mu}_{nit'''}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^A_{nt'''+2}, \qquad (A.75)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\mu}^A_{nit'''} &= \frac{\beta}{\nu}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^A_{it'''+1}-\frac{\beta}{\nu}\sum_{m=1}^N\bar{\mu}_{nmt'''}\beta\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{v}^A_{mt'''+1}, \qquad (A.76)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^A_{nt'''+1} &= \sum_{i=1}^N\bar{\psi}^A_{int'''+1}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\mu}^A_{int'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^A_{it'''}\right), \qquad (A.77)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^B_{nt'''+1} &= \sum_{i=1}^N\bar{\psi}^B_{int'''+1}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\mu}^B_{int'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^B_{it'''}\right), \qquad (A.78)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}_{nt'''+1} &= \frac{\bar{l}^A_{nt'''+1}}{\bar{l}_{nt'''+1}}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^A_{nt'''+1}+\frac{\bar{l}^B_{nt'''+1}}{\bar{l}_{nt'''+1}}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}^B_{nt'''+1}, \qquad (A.79)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\lambda}_{nit'''} &= -\theta\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{it'}-\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{P}_{nt'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\kappa}_{nit'''}\right)+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{z}_{it'''}, \qquad (A.80)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{P}_{nt'''} &= \sum_{i=1}^N\bar{\lambda}_{int'''}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{it'}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\kappa}_{nit'''}-\frac{1}{\theta}\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{z}_{it'''}\right), \qquad (A.81)\\
\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{nt'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}_{nt'''} &= \sum_{i=1}^N\frac{\bar{\lambda}_{int'''}\bar{w}_{nt'''}\bar{l}_{nt'''}}{\bar{w}_{it'''}\bar{l}_{it'''}}\left(\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{\lambda}_{int'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{w}_{it'''}+\mathbb{E}^A_t\mathbb{E}^B_{t'}\hat{l}_{it'''}\right) \qquad (A.82)
\end{aligned}$$

Intuitively, as A anticipates B believes that A will be convinced, relevant to A's decision today is A's belief about B's belief (second-order beliefs); A's belief about B's belief about A's belief no longer has an independent impact. Note also that these equations take the same form as those characterizing A's belief about B's future migration decisions ((A.48) to (A.55)), and these two systems of equations can be satisfied by the same solutions. Among other things, this implies that $\forall t'' > t'$, $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int''}^B = \mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int''}^A$— type A agents think that type B agents believe in the future A will make the same decision as B. We can replace $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int'''}^A$ in equations (A.48) to (A.55) with $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int'''}^B$ for all $t''' > t$.

Given $\hat{\mu}_{int}^B$ and $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{z}_{it'''}$ (and $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\kappa}_{nit'''}$ in the case of stochastic trade costs), type A's decision in period t is characterized by the solution to the joint system of equations (A.23)-(A.38) and (A.48)-(A.55) for $t > t$ and $t'' > t'$. Following the same rationale, type B's decision can be characterized by the set of equations analogous to (A.23)-(A.38) and (A.48) to (A.55).

The following algorithm solves type A's problem in period t.

i. Given a guess of $\hat{\mu}_{int'}^B$, guess the expected wage and price on A's expected path, $\mathbb{E}_t^A \hat{w}_{nt'}$ and $\mathbb{E}_t^A \hat{P}_{nt'}$, respectively, and solve for A's decision in t and expected decision in $t' > t$ using equations (A.23)- (A.24) and (A.31)-(A.32). As we are conditioning on wages and prices, in doing so, B's future decisions are not needed. We label the solution to this A's decision in the *main branch*. In the case illustrated in Panel (c) of Figure 3, these are the outcomes along the solid blue lines.

ii. Given the outcome from step (i), solve the sub-branch problem described by equations (A.48) to (A.55) sequentially for all $t' = t+1, t+2, ..., T$ to obtain A's belief on B's decision in each t' for future periods $t'' \geq t$, e.g., $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int''}^B$ (i.e., the orange and blue dashed lines in Panel (c) of Figure 3).

    The input to the system of equations characterizing the problem in period t' is $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{z}_{it''}$, which is given outside this problem; the belief on initial (t') period labor allocations $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{l}_{nt'}^A$, which by definition is equivalent to $\mathbb{E}_t^A \hat{l}_{nt'}^A$; and the output of step (i). Among the output in period t' is $\mathbb{E}_t^A \mathbb{E}_{t'}^B \hat{\mu}_{int'}^B$, A's belief about B's migration decision in t', which determines the initial allocation of type B in the $t'+1$ problem.

    Thus we can solve the problem sequentially to obtain A's belief about B's migration, delivering the allocation of B over the main branch $\mathbb{E}_t^A \hat{l}_{int'}^B$.

iii. Plug $\mathbb{E}_t^A \hat{l}_{int'}^A$ and $\mathbb{E}_t^A \hat{l}_{int'}^B$ into the static trade equilibrium in each period along the main branch and solve for the updated $\mathbb{E}_t^A \hat{w}_{nt'}$ and $\mathbb{E}_t^A \hat{P}_{nt'}$.

iv. Iterate on steps (i) to (iii) until convergence.

The procedure explained above solves type A's problem given B's current migration decision in t. Following the same logic, we can solve type B's problem given type A's migration decision today. As in the case of Assumption 1, for the solution to be an equilibrium, these two migration decisions need to be consistent with each other. To find such a fixed point, we iterate on A's and B's problems until convergence.

A.4.4 Assumption 3

Under Assumption 3, agents' decisions are guided by their own (and the other type's) beliefs today, and the belief to which both types *think* they will converge. Suppose we are now solving for the decision of agents in period t. We consider two formalizations of Assumption 3.

1. Type g agents in t form beliefs about events in $t+1$ using $f^g(z_{t+1}|z^t)$ and about events in $t' > t+1$ using $\tilde{f}(z_{t'}|z^t)$, where $\tilde{f}$ is the belief to which agents converge.

2. Type g agents in t form beliefs using $f^g(z_{t+1}|z^t)$, but they *think* that in t' and afterward their beliefs about future will both be governed by $\tilde{f}(z_{t'+1}|z^{t'})$.

Two comments are in order. First, in both formulations, $\tilde{f}$ can be flexible. For example, it can be specialized to f^A or f^B, or their weighted average, which corresponds to a scenario in which agents believe that in the future they will meet in the middle.

Second, to see the difference between these two formalizations, note that under the first formalization, type g agents' belief in t about a history up to t', which we denote by $\tilde{f}^g$, is

$$\tilde{f}^g(z^{t'}|z^t) = \begin{cases} f^g(z^{t'}|z^t), \text{for } t' = t+1 \\ \tilde{f}(z^{t'}|z^t), \text{ for } t' > t+1 \end{cases} \tag{A.83}$$

Under the second formalization, agents' belief in t about a history up to t' is the same as in the benchmark setup, i.e., $f^g(z^{t'}|z^t) = \prod_{j=1}^{t'-t} f^g(z_{t+j}|z^{t+j-1})$. However, in making decisions in period t, agents *think* that in $t' \geq t+1$, their future beliefs will be governed by $\tilde{f}(z_{t'+1}|z^{t'})$.[7]

First formulation. Under the first formulation, in making decisions in period t, agents use $f^g(z_{t+1}|z^t)$ to form beliefs about outcomes in $t+1$ and use $\tilde{f}(z_{t'}|z^t)$ to form beliefs about outcomes in $t+2$ and beyond. We use $\tilde{f}^g$, defined in equation (A.83), to denote the effective belief of group g under this assumption.

An immediate implication of the assumption is that, for all g and $g' \neq g$, $\tilde{f}^g(z^{t'}|z^t) = \tilde{f}^{g'}(z^{t'}|z^t)$. We denote the expectation made under $\tilde{f}^g(z^{t'}|z^t)$ by the expectation operator $\tilde{\mathbb{E}}_t^g$. Then we have

$$\tilde{\mathbb{E}}_t \hat{z}_{t''} = \mathbb{E}_t^g \hat{z}_{t''} = \mathbb{E}_t^g \mathbb{E}_{t'}^{g'} \hat{z}_{t''} \text{ for } t' \geq t+1, \text{ and } t'' \geq t'+1. \tag{A.84}$$

This problem can be solved as a modified version under Assumption 1. Observe first that type A's decision in period t can be characterized up to the first order by the system of equations (A.23)-(A.38), with the only modification being that $\mathbb{E}_t^A$ is replaced by $\tilde{\mathbb{E}}_t^A$. Similarly, the system of equations that characterizes $\tilde{\mathbb{E}}_t^A \hat{\mu}_{int'}^B$— which appears in equation (A.34)—is modified equations (A.57)-(A.64), with the modification being that $\mathbb{E}_t^A$ is replaced by $\tilde{\mathbb{E}}_t^A$. These two modified systems of equations take the same form for periods $t' \geq t+1$. Therefore, $\tilde{\mathbb{E}}_t^A \hat{\mu}_{int'}^B$ in equation (A.34) can be replaced by $\tilde{\mathbb{E}}_t^A \hat{\mu}_{int'}^A$ and we can solve equations (A.23) to (A.38) for $\hat{\mu}_{nit}^A$ decision in t, taking as given $\hat{\mu}_{nit}^B$. Intuitively, in the eyes of type A, both types will have converged in belief in $t' \geq t+1$, so A's belief in t about B's decision in t' is the same as A's belief about A's own decision in t'.

Following the same argument, we can solve an analogous system of equations for $\hat{\mu}_{nit}^B$, taking as given $\hat{\mu}_{nit}^A$. Iterating on this process until convergence delivers the solution to agents' problem in period t under this formalization of Assumption 1.

Second formulation. Consider agents making decisions in period t. We define the expectation operator used by type g as $\tilde{\mathbb{E}}_t^g$. The second formulation implies that for $g \neq g'$, and $\forall t'' > t' > t$,

$$\begin{gathered} \tilde{\mathbb{E}}_t^g z_{t'} \neq \tilde{\mathbb{E}}_t^{g'} z_{t'}, \\ \tilde{\mathbb{E}}_t^g \tilde{\mathbb{E}}_{t'}^{g'} z_{t''} = \tilde{\mathbb{E}}_t^g \tilde{\mathbb{E}}_{t'}^g z_{t''} = \tilde{\mathbb{E}}_t^{g'} \tilde{\mathbb{E}}_{t'}^g z_{t''} = \tilde{\mathbb{E}}_t^{g'} \tilde{\mathbb{E}}_{t'}^g z_{t''} \neq \tilde{\mathbb{E}}_t^g z_{t''} \text{ or } \tilde{\mathbb{E}}_t^{g'} z_{t'}, \end{gathered} \tag{A.85}$$

where the inequality in the second line reflects time inconsistency in agents' beliefs.

This formulation can be solved as a modified version of Assumption 2. To build towards an

[7] Agents' belief display time-inconsistency. For example, for $t'' > t' > t$, $\tilde{f}(z^{t''}|z^t) \neq f^g(z^{t'}|z^t) \cdot \tilde{f}(z^{t''}|z^{t'})$.

algorithm of solving the model under this formulation, first note that to solve the equations characterizing A's decision in period t using equations (A.23) to (A.38), we need to solve for $\tilde{\mathbb{E}}_t^A \hat{\mu}_{int'}^B$. The system of equations that characterizes $\tilde{\mathbb{E}}_t^A \hat{\mu}_{int'}^B$, equations (A.48) to (A.55), continues to hold with the following modification: $\mathbb{E}_t^A \mathbb{E}_{t'}^B$ is replaced with $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_t^B$.

Moreover, under Assumption 3, third- and higher-order beliefs specialize to second orders. Therefore, the operator $\mathbb{E}_t^A \mathbb{E}_{t'}^B \mathbb{E}_{t''}^A$ in the equations characterizing A's belief about B's belief about A's future decisions, equations (A.67) to (A.74), can be replaced by $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_{t'}^B$. This system of equations now takes the same form as the modified equations (A.48) to (A.55) described in the preceeding paragraph, which means the solution to one set of the equations is also the solution to the other. Intuitively, as type A agents think that type B agents believe in the future, A will make the same decision as B—both according to $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_{t'}^B$—we can replace $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_{t'}^B \hat{\mu}_{int'''}^A$ in equations (A.48) to (A.55) with $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_{t'}^B \hat{\mu}_{int'''}^B$ for all $t''' > t$.

Given $\hat{\mu}_{int}^B$, $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_{t'}^B \hat{\kappa}_{nit'''}$ and $\tilde{\mathbb{E}}_t^A \tilde{\mathbb{E}}_{t'}^B \hat{z}_{it'''}$, type A's decision in period t is characterized by the solution to the modified joint system of equations (A.23) to (A.38) and (A.48) to (A.55) for $t' > t$ and $t'' > t'$. Following the same argument, analogous equations characterize $\hat{\mu}_{int}^B$ given $\hat{\mu}_{int}^A$, $\tilde{\mathbb{E}}_t^B \tilde{\mathbb{E}}_{t'}^A \hat{\kappa}_{nit'''}$, and $\tilde{\mathbb{E}}_t^B \tilde{\mathbb{E}}_{t'}^A \hat{z}_{it'''}$. This problem is parallel to that described in Section A.4.3, and the algorithm described there applies here.

A.5 Proof of Proposition 5

In this subsection, we show that the second-order accurate solution method, heterogeneous beliefs (under the assumptions on higher-order beliefs), and ex-post analysis can be tractably combined.

A.5.1 Second-Order Solution in Ex-Post Evaluations

We first explain how to obtain second-order counterfactual/perfect foresight outcomes in ex-post evaluations, where the observed allocation is shaped by agents making decisions under evolving and uncertain beliefs. Without loss of generality, assume researchers observe the realizations over $t = 1, 2, ..., T$, denoted by $\tilde{x}_{it}$, with $x_{it} \in \{v_{it}, P_{it}, w_{it}, \mu_{int}, \lambda_{int}, l_{nt}\}$.[8]

We extend the algorithm described in Section 2.5. Specifically, in steps (i) and (iii) of the algorithm, instead of solving for the expected outcomes with first-order accuracy using Proposition 1, use Proposition 3 to solve for the expected outcomes with second-order accuracy.[9] At steps (ii) and (iv) of the algorithm, append the path obtained in steps (i) and (iii) with actual outcomes in the economy. Note that because these actual outcomes are exact by definition and because the deviations are solved with second-order accuracy, the output of the algorithm, the expected paths based on the beliefs in different periods, are second-order accurate.

We can then obtain the perfect foresight path for each period by approximating around the expected path using Proposition 3, taking the expected values (with second-order accuracy) and covariances as inputs. We can then approximate around either the perfect foresight or the belief path to obtain counterfactual paths with second-order accuracy by using Proposition 3.

A.5.2 Second-Order Solution with Heterogeneous Beliefs

We now show how to obtain second-order accurate solutions under each of the three assumptions about the structure of heterogeneous beliefs.

[8] When the data ends in $T' < T$, the researcher can first extend the data using time-differenced model under additional assumption on how fundamentals evolve between T' and T.

[9] This involves simulating deviations and solving for the outcomes under these simulated deviations. Then calculate expected second-order moments, and use these to calculate the expected outcomes with second-order accuracy.

Assumption 1. We have shown that the first-order approximation to the solution of the period t problem is characterized by equations (A.23)-(A.38) for A, taking B's decision at period t as given, and analogous equations for B, taking A's decision in t as given.

To achieve second-order accuracy in solving this problem, we can expand equations (A.23)-(A.38) for both types of agents to second order and apply Assumption 1, which implies that A's belief about B's future decisions is the same as A's belief about A's own future decisions; and vice versa for B. The result is essentially equation (A.19), with $\mathbb{E}_t$ being replaced by the type-specific expectation operator $\mathbb{E}_t^g$ for $g \in \{A, B\}$. We call this the modified equation (A.19).

The following algorithm solves the problem up to second order:

i. Solve for the first-order solution to agents' decision in period t following the algorithm described in Section A.4.2.

ii. Simulate S paths of fundamentals for $t+1, t+2, ..., T$ using A's belief $f^A(z_{t+1}|z^t)$. Start at the labor allocation in period $t+1$ that is implied by the $\hat{\mu}_{nit}^A$ and $\hat{\mu}_{nit}^B$ calculated in step (i), and solve the sequential outcomes on $\{t+1, t+2, ..., T\}$. In the eyes of type A in period t, in $t+1$ and beyond, all agents make the same decisions as if they have the belief of A. Therefore, in their view, the evolution of the economy under each realization of fundamentals would be as if all agents have the same belief. This problem can be solved using Proposition 1.

   The solution has first-order accuracy because first, the initial allocation of the economy in $t+1$, based on $\hat{\mu}_{nit}^A$ and $\hat{\mu}_{nit}^B$, has first-order accuracy; and second, Proposition 1 delivers first-order accuracy conditional on the starting point.

   Use the simulated paths to calculate the second-order moments that appear in the modified equation (A.19) for A with $\mathbb{E}_t^A$ as the expectation operator. These second-order terms have second-order accuracy following Proposition 3.

iii. Carry out step (ii) for type B agents.

iv. Iterate on the modified equation (A.19) for A's and B's decisions in period t, taking the second-order terms constructed in steps (ii) and (iii) as constants.

   The output of this step is $\hat{\mu}_{nit}^A$ and $\hat{\mu}_{nit}^B$ that have second-order accuracy.

v. Use $\hat{\mu}_{nit}^A$ and $\hat{\mu}_{nit}^B$ to obtain the initial allocation in $t+1$.

vi. Repeating steps (i)–(v) for $t+1, t+2, .., T-1$ delivers an evolution of the economy under second-order approximation.

Remark. Note that in the above algorithm, in all steps that involve simulations, we only need to solve a homogeneous belief problem for each type. Heterogeneous beliefs enter this algorithm only in steps (i) and (iv) when we solve for agents' actual decisions in period t. Aside from these two fixed points—one first-order, one second-order—the rest of the problem is as tractable as the second-order solution for the homogeneous belief problem described in Proposition 3.

Assumption 2. The algorithm we develop for solving the model up to second-order under Assumption 2 is more computationally demanding, but conceptually it follows from the first-order algorithm described in Section A.4.3.

i. Simulate S future *trees* (not just paths) that correspond to A's problem in t. That is, simulate a set of paths of fundamentals with the 'main branch' from $f^A(z^T|z^t)$ (solid lines in Figure 3c) and the off-branch 'limbs' from $f^B(z^T|z^{t'}) \cdot f^A(z^{t'}|z^t)$. We call each such simulation a tree. The 'limbs' are simulated according to $f^B(z^T|z^{t'}) \cdot f^A(z^{t'}|z^t)$ because A in t thinks that in $t+1$ and beyond, B will believe that A and B both make decisions according to B's beliefs.

ii. For each simulation, solve the evolution of the economy from $t+1$ to T using the algorithm described in Section A.4.3, replacing $\mathbb{E}_t^A$ with the simulated 'main branch' and $\mathbb{E}_t^A \mathbb{E}_{t'}^B$ with the simulated 'limbs' at t' for all $t' \geq t$. Doing so for all simulations delivers a number of possible future outcomes from the perspective of type A in period t .

   Use these simulations to calculate the second-order terms that appear in the second-order extensions of the equations that appear in A's period t problem described in Section A.4.3.

iii. Follow steps (i) and (ii) to construct the second-order terms that appear in B's period t problem analogously.

iv. Solve the system of equations from steps (ii) and (iii) jointly for the remaining first-order terms. One way of approaching the problem is to take $\hat{\mu}_{nit}^B$ as given and solve for $\hat{\mu}_{nit}^A$ using the equations from (ii); do the same for B using the equations from (iii). Iterate between these two steps until convergence.

v. The above process delivers $\hat{\mu}_{nit}^A$ and $\hat{\mu}_{nit}^B$, which shifts the economy in $t+1$. Repeating the process for $t+1, t+2, ..., T$ delivers the evolution of the economy.

Assumption 3. As described in Section A.4.4, Assumption 3 can be formulated in two different ways. The first formulation can be solved up to first order via the modified algorithm for Assumption 1; the second formulation, via the modified algorithm for Assumption 2.

To obtain the second-order solution to the evolution of the economy, one can follow the algorithms described above for Assumption 1 and 2, respectively. The main difference is that in simulating the path of fundamentals, one needs to use $\tilde{f}^A$ and $\tilde{f}^B$ rather than f^A and f^B.

A.5.3 Heterogeneous Beliefs in Ex-Post Studies

We now consider counterfactuals in ex-post evaluations when agents have heterogeneous beliefs.

Suppose the researcher observes the evolution of outcomes from $t = 1$ to T. The economy features heterogeneous beliefs and the researcher is interested in counterfactual questions such as how the economy would evolve if agents held different beliefs from the ones that generate the data or if the fundamental productivity processes have a different realization.

Recall that in a homogenous belief setting, a key aspect of ex-post analysis is to disentangle the role of beliefs from that of time-varying migration costs in shaping the observed allocation. In settings with heterogeneous beliefs, this identification problem is more involved since one also needs to disentangle the roles of *heterogeneous* migration costs and *heterogeneous* beliefs. In other words, if the researcher observes different migration behaviors between people holding different beliefs, part of the difference may be rationalized by their facing different migration costs.

Depending on the data available to the researcher, one may rationalize the data under different assumptions on how migration costs differ. Below we discuss how to proceed under two polar cases: one in which the researcher has all the degrees of freedom to allow migration costs to vary arbitrarily between people with different beliefs, and one in which the researcher has none.

Stronger data requirement. We start with the case in which the researcher observes all past decisions by belief type (i.e., μ_{nit}^g, $g \in \{A, B\}, t = 1, ..., T$). In this case, the researcher can allow migration costs to be time-varying and differ between people holding different beliefs. Intuitively, although in principle belief differences can explain all differences in migration behaviors, once beliefs are disciplined using external measures, the remaining differences in migration behaviors between belief types observed in the data can be attributed to migration costs.

We can extend Proposition 4 to accommodate different migration costs between the types by approximating around a deterministic sequential path with homogeneous belief and two types of agents that differ in their migration costs. The approximation to the market clearing condition is the same as in Proposition 4. The approximation to migration decisions is type-specific: i.e., A's decision (and B's belief about A's decision, A's belief about B's belief about A's decision and so on) is approximated around A's decision in the perfect foresight path; B's decision (and A's belief about B's decision and so on) is approximated around B's decision in the perfect foresight path. As this approximation is within a period and agent type, time-varying heterogeneous migration costs will be differentiated out.

The algorithm below generalizes the one in Section 2.5 to accommodate heterogeneous beliefs:

i. Solve for the expected outcomes in period T according to agents' belief in $T-1$, i.e., $\mathbb{E}^g_{T-1}(w_{nT})$, $\mathbb{E}^g_{T-1}(P_{nT})$, $\mathbb{E}^g_{T-1}(\lambda_{nT})$ for $g \in \{A, B\}$, using (A.28)-(A.30) for $t = T-1$, by approximating around the actual outcome in period T.

ii. Append the outcome from step (i) with A and B's actual migration in period $T-1$, μ^A_{inT-1} and μ^B_{inT-1}, and the outcomes from the static trade equilibrium in $T-1$, to obtain the solution to the joint system of equations at period $T-1$ defined in Section A.4. Note that as $T-1$ is the second-to-the-last period, second- and higher-order beliefs do not matter for decisions in $T-1$. Therefore, solving the problem in period $T-1$ around a perfect foresight path is essentially solving equations in Proposition 1 extended to two types of agents.

iii. Solve for the $\mathbb{E}^g_{T-2}(w_{nT-1})$, $\mathbb{E}^g_{T-2}(w_{nT})$, $\mathbb{E}^g_{T-2}(P_{nT-1})$, $\mathbb{E}^g_{T-2}(P_{nT})$, and $\mathbb{E}^g_{T-2}(\mu^A_{nT-1})$ for $g \in \{A, B\}$ by approximating around the outcome in the previous step. This amounts to solving (A.23)-(A.38) for $t = T-2$ with the approximation points being $\mathbb{E}^g_{T-1}(x_t)$, $x \in \{P_n, \mu^g_{ni}, l^g_i, w_n, \lambda_{ni}\}$ for $t = \{T-1, T\}$. The input is the difference in the first- and second-order beliefs between $T-2$ and $T-1$ about fundamentals in $\{T-1, T\}$ (for A and B, respectively). Solving this under each of the three assumptions on higher-order beliefs can be done analogously to the process explained in the proof of Proposition 4. The output of this step is in deviations. These deviations, when multiplied by the level outcomes from the previous step, give the expected value of outcomes according to beliefs at $T-2$.

iv. Append the above with the actual migration decisions of A and B and the outcomes from the static trade equilibrium in period $T-2$, which delivers the solution to the problem of A and B in $T-2$.

v. Iterate on steps (iii) and (iv) from $T-3$ backward until $t = 1$.

Weaker data requirement. We now discuss a case in which the researcher does not have the degree of freedom to allow migration costs to vary between types.

Suppose the researcher observes the total (the sum across belief types) migration flows across locations over $t = 1, .., T$. In addition, the researcher has access to the cross-sectional distribution of the population by belief in *just one* period. We show that the model is just identified if agents with different beliefs have the same migration costs (which can still be time-varying). Thus, if the researcher allows migration costs to differ by type, the model will be under-identified.[10]

Below, we develop a recursive algorithm to recover belief paths for each type, assuming that the researcher observed the distribution of population by belief in period T. We show that, given the belief structure, there is a unique way of rationalizing the observed migration patterns and the distribution of agents by belief in T when agents face the same migration cost. Finally, we discuss how to modify the algorithm if the distribution is available for $T' < T$.

[10]On the other hand, if the researcher has access to more than one cross-sectional distribution of the population by belief type, the model will be over-identified. We discuss this scenario in the end of this section.

i. Solve for $\mathbb{E}^{g}_{T-1}(w_{nT})$, $\mathbb{E}^{g}_{T-1}(P_{nT})$, $\mathbb{E}^{g}_{T-1}(\lambda_{nT})$ for $g \in \{A, B\}$ using (A.28)-(A.30) for $t = T-1$ by approximating around the actual outcome in period T.

ii. Use output from step (i) to derive the *difference* in the migration decisions of A and B. The intuition is that the difference between A's and B's migration decisions in period $T-1$ is determined entirely by their beliefs about T outcomes, which we have derived in the previous step. Thus, the difference between $\mathbb{E}^{A}_{T-1}U(w_{nT}/p_{nT})$ and $\mathbb{E}^{B}_{T-1}U(w_{nT}/p_{nT})$ contains information about the difference in migration behavior between A and B.

   To implement this idea, linearize A's decision around B's decision (which is an unobserved $N \times N$ matrix, denoted by μ^{B}_{niT-1} for now), which expresses A's decisions as a function of μ^{B}_{niT-1} and the difference between $\mathbb{E}^{A}_{T-1}U(w_{nT}/p_{nT})$, and $\mathbb{E}^{B}_{T-1}U(w_{nT}/p_{nT})$. Note also that the weighted sum of μ^{B}_{niT-1} and μ^{A}_{niT-1} should be equal to the total $N \times N$ migration matrix in the data. Let variables with a tilde indicate the observed allocation. This yields the following system of equations:

$$\begin{aligned}
\hat{\mu}^{A}_{nmT-1} &= \frac{\beta}{\nu}\sum_{i=1}^{N}\Big(\mathbb{1}(i=m)-\mu^{B}_{niT-1}\Big)[\mathbb{E}^{A}_{T-1}U(w_{iT}/p_{iT})-\mathbb{E}^{B}_{T-1}U(w_{iT}/p_{iT})] \qquad \text{(A.86)}\\
\mu^{A}_{nmT-1} &\equiv \exp(\hat{\mu}^{A}_{nmT-1})\mu^{B}_{nmT-1}\\
\tilde{\mu}_{nmT-1}\tilde{l}_{nT-1} &= \mu^{A}_{nmT-1}l^{A}_{nT-1}+\mu^{B}_{nmT-1}l^{B}_{nT-1}\\
\tilde{l}^{A}_{nT} &= \sum_{i=1}^{N}\mu^{A}_{inT-1}l^{A}_{iT-1}\\
\tilde{l}^{B}_{nT} &= \sum_{i=1}^{N}\mu^{B}_{inT-1}l^{B}_{iT-1}
\end{aligned}$$

   where the right-hand side of the first equation follows from T being the last period; the third equation requires that total migration is consistent with the decomposition by belief type.

   Combining the first two equations to eliminate $\hat{\mu}^{A}_{nmT-1}$, the remaining system of equations then has, up to normalization, $N \times N \times 2 + N \times 2$ unknowns ($\mu^{A}_{nmT-1}, \mu^{B}_{nmT-1}, l^{A}_{iT-1}, l^{B}_{iT-1}$) and the same number of equations. This shows that, given the belief structure, there is a unique way of rationalizing the observed migration patterns and the distribution of agents by belief in T when agents face the same migration cost.

   This step also delivers the distribution of agents by belief in period $T-1$. We can use these to recover agents' distribution recursively backward, as described below.

iii. Append the output from step (i) with agents' migration decision by type in $T-1$, obtained in step (ii), and the outcome of the static trade equilibrium in $T-1$.

iv. Use the output from step (iii) to recover the belief paths in period $T-2$ by agents' belief type about outcomes in period $T-1$ and beyond, following step (iii) of the algorithm for heterogeneous belief case under the stronger data requirement.

v. Step (iv) also delivers $\mathbb{E}^{g}_{T-2}v^{g'}_{nT-1}$ for $g, g' \in \{A, B\}$. Use the value difference between A's and B's beliefs in step (iv) to derive the migration decisions analogous to equations (A.86). For example, the deviation of A's migration decision from B's in $T-2$ should satisfy

$$\hat{\mu}^{A}_{nmT-2} = \frac{\beta}{\nu}\sum_{i=1}^{N}\Big(\mathbb{1}(i=m)-\mu^{B}_{niT-2}\Big)\cdot[\mathbb{E}^{A}_{T-2}v^{A}_{iT-1}-\mathbb{E}^{B}_{T-2}v^{B}_{iT-1}].$$

Solving this system of equations delivers μ^{g}_{nmT-2} and l^{g}_{nT-2} for $g \in \{A, B\}$.

Appending these, as well as the outcome of the static trade model in $T-2$, delivers agents' belief path in period $T-2$.

vi. Iterate on steps (iii)–(iv) until $t = 1$.

In the above algorithm, we assume that the researcher observes the distribution of the population by their belief type in T. If instead the researcher observes the distribution in $T' \neq T$, we can nest the above algorithm within an outer loop in which we guess a distribution by belief type in period T, iterate on the above algorithm for $T-1, T-2, ..., T'$, and check whether the distribution of the population by belief type in T' is consistent with the data. We can iterate on this process in the outer loop until convergence.

Finally, if researchers have more than one cross-sectional of the distribution of belief types, the model is over-identified. In such scenarios, researchers can attempt to match the available distributions in different periods jointly so all information is used; alternatively, the researcher can use a subset of the distributions, reserving the rest for validation.

Appendix B Applications

B.1 Multi-Sector Extension

Our quantitative applications use an extended model. We describe the main aspects of the extension below, relegating the details to the supplementary appendix. We denote by N the number of locations, indexed by $\{n, i, m, o, h\}$, and by J the number of sectors, indexed by $\{j, k, s\}$.

Preference. The consumption in region n sector j at time t is defined by

$$C^{nj}_t = w^{nj}_t / P^n_t,$$

where P^n_t is the ideal price index in region n at time t given by

$$P^n_t \equiv \prod_{j=1}^{J} (P^{nj}_t / \alpha^{nj})^{\alpha^{nj}}. \tag{B.1}$$

Here P^{nj}_t is the price of the sectoral final good in sector j and region n at time t, defined below, and α^{nj} is the final expenditure share in sector j and region n with $\sum_{j=1}^{J} \alpha^{nj} = 1$.

Production. As in Eaton and Kortum (2002), sectoral final goods are aggregated from a continuum of intermediate inputs in the same sector by a representative producer. The production of intermediate goods in each sector combines labor and the sector's own final goods as inputs, according to production technology

$$q^{nj}_t(\omega) \quad = \omega \left(l^{nj}_t(\omega) \right)^{\gamma^{nj}} \left(M^{nj}_t(\omega) \right)^{1-\gamma^{nj}}, \tag{B.2}$$

where $q^{nj}_t(\omega)$ is the output quantity; ω is the productivity of the producer, which is drawn from a Frechet distribution with location parameter z^{nj}_t; $l^{nj}_t(\omega)$ is the use of labor; $M^{nj}_t(\omega)$ is the use of the sectoral final good in j; and γ^{nj} is the share of value added in intermediate goods production in sector j and region n.

The price of sectoral final goods in location n sector j after a history z^t is given by

$$P_t^{nj}(z^t) \quad \propto \quad \left[\sum_{i=1}^{N}\left((w_t^{ij}(z^t))^{\gamma_{ij}}(P_t^{ij}(z^t))^{1-\gamma_{ij}}\kappa_t^{nj,ij}\right)^{-\theta} z_t^{ij}\right]^{-1/\theta}. \tag{B.3}$$

Migration decision. In each period, workers choose both location and sector, subject to the migration costs of moving from the current location-sector nj to a destination location-sector ik, denoted by $m^{nj,ik}$. The value of location-sector nj for a representative agent there is

$$v_t^{nj}(z^t) \quad = \quad U\left(C_t^{nj}(z^t))/P_t^n(z^t)\right) + \nu \log\left[\sum_{i=1}^{N}\sum_{k=1}^{J}\exp\left(\beta\mathbb{E}[v_{t+1}^{ik}(z^{t+1})|z^t] - m^{nj,ik}\right)^{1/\nu}\right]. \tag{B.4}$$

The fraction of households that relocate from market nj to ik is given by

$$\mu_t^{nj,ik}(z^t) \quad = \quad \frac{\exp\left(\beta\mathbb{E}[v_{t+1}^{ik}(z^{t+1})|z^t] - m_t^{nj,ik}\right)^{1/\nu}}{\sum_{m=1}^{N}\sum_{h=1}^{J}\exp\left(\beta\mathbb{E}[v_{t+1}^{mh}(z^{t+1})|z^t] - m_t^{nj,mh}\right)^{1/\nu}}. \tag{B.5}$$

B.2 Climate Change Application

B.2.1 Geographic and Time Coverage

List of countries and regions. We calibrate the model to the 50 U.S. states and 28 other countries/regions including a synthetic rest of the world. These countries/regions include Australia, Austria, Balkans (aggregating Bulgaria, Croatia, and Greece), Baltic States (aggregating Estonia, Latvia, and Lithuania), Benelux (Belgium, Netherlands, Luxembourg), Brazil, Canada, Switzerland, China, Central Europe (aggregating Poland, Czech Republic, Slovakia, Slovenia, and Hungary), Germany, Spain, France, United Kingdom, Indonesia, India, Ireland, Italy, Japan, South Korea, Mexico, Nordic Countries (aggregating Denmark, Finland, Norway, and Sweden), Portugal, Romania, Russia, Turkey, Taiwan, and the rest of the world. Regarding the 50 U.S. States, the District of Columbia is aggregated into Virginia.

Definition of the five broad U.S. regions. We report some of the results for five broad U.S. regions: Northeast (NE), Southeast (SE), Midwest (ME), Southwest (SW), and West (WE). These regions are defined as follows:

- Northeast: Connecticut, Delaware, Maine, Maryland, Massachusetts, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, and Vermont.
- Southeast: Alabama, Arkansas, Florida, Georgia, Kentucky, Louisiana, Mississippi, North Carolina, Tennessee, South Carolina, Virginia, and West Virginia.
- Midwest: Iowa, Illinois, Indiana, Kansas, Michigan, Missouri, Ohio, Wisconsin, Minnesota, Nebraska, North Dakota, and South Dakota.
- Southwest: Arizona, New Mexico, Oklahoma, and Texas.
- West: Alaska, California, Colorado, Hawaii, Idaho, Montana, Nevada, Oregon, Utah, Washington, and Wyoming.

Time horizon. We treat 2013–2014 as the starting point of the model and collect from various data sources the information necessary for constructing this initial economy. Our analysis focuses on the years 2014–2100. We treat each period in the model as two years. Consistent with this treatment, we set $\beta = 0.92$, corresponding to an annual discount rate of 4%; we also use two-year migration flows, constructed based on the job flow information provided by the Longitudinal Employment History Database, described below.

B.2.2 Data Construction

To construct the initial economy, we need information on migration, trade, and production around 2014. To solve for the evolution of the economy over 2014–2100, we also need temperature projection (mean and variance) and the distribution of climate skeptics. This subsection explains the sources of information and data construction in detail.

We obtain data on production, value added, and trade flows at the country-sector level from the World Input-Output Database (WIOD). We use the WIOD to construct a few pieces of information, aggregating all information into three broad sectors: agriculture, manufacturing, and service.

Sectoral shares in final consumption. To calculate sectoral shares in final consumption, α^{nj}, we sum the final expenditures on sector j goods by households, nonprofit organizations, and governments in each country n and divide the sum by the total final absorption of n country in the WIOD. We do this for both foreign countries and the United States, assuming that α^{nj} is common across U.S. states.

Value-added share in intermediate goods production. We compute the share of value added in the gross output γ^{nj} and the intermediate consumption share $1 - \gamma^{nj}$ by dividing total intermediate input uses in sector j of country n by the gross output of the sector-country cell. Again, γ^{nj} are identical across U.S. states but differ by country.

Production and absorption. The production and absorption of sectoral final goods for the United States as a whole and for foreign countries are obtained directly from the WIOD. To construct the absorption and production for individual U.S. states, we use the state-sector level value added from the U.S. Bureau of Economic Analysis (BEA), and the sectoral shares in consumption α^{nj} and value-added shares in sectoral production γ^{nj}. We combine these data with the model-implied market clearing conditions for local agriculture, manufacturing, and service final goods to calculate both absorption and production, which we then scale so that for the United States as a whole, the aggregate statistics are consistent with the WIOD.

International and inter-state trade. Trade shares between non-U.S. countries are directly computed using the WIOD.[11] Since our model is at the state-sector level within the United States, we need to construct inter-state trade and trade between U.S. states and other countries. We do this by supplementing the WIOD with additional information, as described below:

i. Trade between U.S. states and other countries

   We use the U.S. trade data provided by the U.S. Census to compute each state's share in U.S. exports to each destination country and U.S. imports from each source country. We then apply these shares to the bilateral trade flows between the United States and each other country in the WIOD to obtain the level of trade flows between U.S. states and other countries.

ii. Trade flows between U.S. states

   (a) Manufacturing: We use 2012 Commodity Flow Survey (CFS) data to obtain the bilateral trade flows across individual U.S. states for the manufacturing sector. We scale the total level of transactions in CFS to match the domestic sales of the manufacturing sector in the United States from the WIOD.

[11] We assume that service is non-tradable so we attribute all service consumption expenditures to local production.

(b) Agriculture: We first estimate the elasticities based on a gravity equation using the inter-state trade flow in manufacturing constructed above. We then impute agricultural trade flows using the estimated elasticities and the absorption and production in the agriculture sector. The gravity specification we use is

$$\ln(X_t^{in,A}) = \beta_1 \ln(X_t^{n,A}) + \beta_2 \ln(Y_t^{i,A}) + \beta_3 \ln(\text{dist}^{in}) + \beta_4 \mathbb{1}(i = n), \tag{B.6}$$

where $X_t^{in,A}$ is the bilateral flow, $\ln(X_t^{n,A})$ is the absorption in the destination, and $Y_t^{i,A}$ is the production in the source country. Absorption and production are constructed as described above. Bilateral distance is aggregated from the county-pair, weighted by population. We scale the level of transactions to match the domestic sales in the U.S. agriculture sector from the WIOD. The estimated coefficients are $\hat{\beta}_1 = 0.786$, $\hat{\beta}_2 = 0.745$, $\hat{\beta}_3 = -1.668$, and $\hat{\beta}_4 = 0.848$.

(c) Service: We assume there is no trade between U.S. states.

Using the trade flows between U.S. states, between U.S. states and foreign countries, and between foreign countries, as constructed above, we construct a bilateral trade *share* matrix.

Employment and migration.

1. **Initial labor allocation** l_0. We construct the initial labor allocation in 2014 (l_0^{nj}) using the employment data across sectors and states obtained from the BEA. We normalize the level so that $\sum_{n=1}^{R} \sum_{j=1}^{J} l_0^{nj} = 1$.

2. **Initial migration share**. The Longitudinal Employer-Household Dynamics Database (LEHD) job-to-job flow data (J2JOD) contains mobility flow for state-sector level quarterly data. We aggregate the quarterly data into the annual level for 2013, defined by job-to-job hires including brief non-employment. Dividing this by the employment in each state sector gives us the share of out-migrants. We treat the residual as stayers.[12]

   In the climate change application, each period is two years. To generate two-year flows, we convert the one-year migration shares to a two-year migration share matrix.

Temperature projections. We construct temperature projections based on the Climate Impact Lab, accessed on September 9, 2022.[13] The Climate Impact Lab provides six cases: two carbon emissions scenarios (RCP 4.5 and RCP 8.5) with three paths for temperature rise for each scenario (i.e., median, 5th percentile, and 95th percentile). In each case, the projection is reported for four broad periods—1986–2005 (historic), 2020–2039, 2040–2059, and 2080–2099—and is available for both foreign countries and individual U.S. states. Our quantification focuses on the RCP 4.5 scenario, which is described by the IPCC as an intermediate scenario with carbon emissions peaking in 2040 and then declining.

Interpolating for yearly frequency. We interpolate the temperature within these time frames linearly to obtain yearly frequency data for the median, 5^{th}, and 95^{th} percentiles in temperature projection. To be specific, we assume the projected temperature in each period is for the middle year of each period (2030, 2050, and 2090). We then linearly interpolate the temperature between those years and 1996 (the middle year between 1986 and 2005).[14] We treat the value of the median path in 2014 from this interpolation as the temperature for 2014.

[12] While LEHD J2JOD data contain information on the stayers, we do not use it since it does not count those who did not change their job.

[13] Climate Impact Lab updated the projection scenarios recently; our raw data are available upon request.

[14] For 2090-2100, we extrapolate from the 2051-2090 path.

Variance profile. We adopt the RCP 4.5 median scenario as the average temperature path and use the median and 5^{th} percentile likelihood values to calibrate $(\sigma^1_{nt})^2$, the variance in the average annual temperature in period t from the perspective of the agent in the first period.[15] Specifically, we choose $(\sigma^1_{nt})^2$ so that the difference between the median and the 5^{th} percentile temperature projections in t is the same as implied by $(\sigma^1_{nt})^2$ under the Gaussian assumption on the temperature path. That is,

$$(\sigma^1_{nt})^2 = \left((\bar{\mathcal{C}}^1_{nt} - \mathcal{C}^{1,5th}_{nt})/1.645\right)^2,$$

where $\mathcal{C}^{1,5th}_{nt}$ denotes the 5^{th} percentile projection for the temperature in region n for period t from today. We assume that the variance profile depends only on the duration of the forecast. That is, the uncertainty perceived in the forecast made in 1996 about 2010 is the same as the uncertainty perceived in 2030 about 2044. Formally, the variance in the annual average temperature in period t' for an agent in period $t < t'$, denoted by $\sigma^t_{nt'}$, is simply $\sigma^1_{nt'-t+1}$: for example, $\sigma^5_{n7} = \sigma^1_{n3}$.

Spatial distribution of climate skeptics. We use data from the 2014 Yale Climate Opinion Survey to calibrate the initial share of climate skeptics in each state, defined as those answering "No" or "Don't know" to the following question: "What do you think: Do you think that global warming is happening?" We define those answering "yes" as believers of climate change. The average share of believers across U.S. states in 2014 is 62.6% and the standard deviation across states is 5.32%.

Sector aggregation. We incorporate three broad sectors—agriculture, manufacturing, and service—and aggregate sector-level information from various sources of data to these three broad sectors. Table B.1 summarizes the aggregation.

Table B.1: List of Sectors and Datasets

Dataset	Agriculture	Manufacturing	Service
WIOD	Agriculture, Forestry, Fishing, Mining (sectors A and B)	Manufacturing (sector C)	The rest
BEA:GDP	Agriculture, forestry, fishing, and hunting Mining, quarrying, and oil and gas extraction	Manufacturing	The rest
BEA:EMPLOYMENT	Farm employment (linecode 70) Forestry, fishing, and related activities Mining, quarrying, and oil and gas extraction	Manufacturing (linecode 500)	The rest
CENSUS (US TRADE ONLINE)	111 Agricultural Products 112 Livestock & Livestock Products 113 Forestry Products, Nesoi 114 Fish, Fresh/chilled/frozen & Other Marine Products 211 Oil & Gas 212 Minerals & Ores	The rest	N/A
CFS	N/A	Manufacturing (Sectors 1–12)	N/A
LEHD	Agriculture, forestry, fishing, and hunting Mining, quarrying, and oil and gas extraction	Manufacturing (Sectors 31–33)	The rest

[15] Note that if we adopt RCP 8.5, which assumes high future emissions and a dramatic expansion of coal use, the uncertainty in the later period would be larger. Similarly, as the 95^{th} percentile scenario implies larger increase in temperature, using the difference between 5^{th} percentile and the median path implies smaller uncertainty. In this sense, our quantification is a conservative estimate of the impact of uncertainty.

B.3 China Shock Application

B.3.1 Geographic and Time Coverage

Regions and sectors. We focus on the same set of countries and (aggregated) regions as in the climate change application, explained in Section B.2.1.

We focus on two sectors, manufacturing and non-manufacturing. For all variables, non-manufacturing is obtained by aggregating the agricultural and service sectors as defined in Table B.1.

Time horizon. Our analysis covers 2000–2099. Each period is treated as a year. Correspondingly, we set $\beta = 0.96$.

B.3.2 Data Construction

In this application, we treat 2000–2014 as the in-sample period for which we construct the evolution of the economy from the data. We treat years after 2014 as out of the sample. Thus, unlike in the climate change application, where only the data from 2014 are used, in this China shock application we need to construct the data for every year between 2000 and 2014. Most of the data we use for the China shock application are the same as those used in the climate change application, so the same data source and procedure apply. However, because some data are not available for all years over 2000–2014, we need to fill in the gaps in the time series in some cases and use alternative data sources in other cases. Below, we explain the *main differences* in data construction in the China shock application and the climate change application.

Migration. As mentioned in Section 4.1, LEHD job-to-job flow data miss some states over 2000–2009. Table B.2 shows the list of missing states in LEHD. We fill in the missing values using migration in Caliendo et al. (2019), which constructs migration flows between states and sectors over this period using data from the American Community Survey. Their data cover 23 sectors, including a non-employment sector. We remove the flows associated with the non-employment sector, normalize the rest of the values to match our data sequence, and then aggregate the sectors in their data into the two aggregate sectors in our model.[16]

Table B.2: States Not Covered by LEHD

Year	States
2000	AL, AZ, AR, KY, MA, MS, NH, WY
2001-2002	AZ, AR, MA, MS, NH, MI
2003	AR, MA
2004-2009	MA

Sectoral production and consumption shares. As the data covers 2000–2014, we construct the final consumption shares α_{nj} and value-added share γ^{nj} in production using the data from the year 2007, the middle of the sample period. The construction process is the same as described in Section B.2.2 of this appendix.

International and inter-state trade. To construct trade flows between U.S. states and foreign countries, and between U.S. states, we proceed as follows:

i. Trade between U.S. states and other countries

   We use each state's share in U.S. output in each sector as the state's share in U.S. exports to each country and imports from each country. We use output instead of imports and exports

[16] In particular, we map sectors 2-13 and sectors 14-23 in Caliendo et al. (2019) to the manufacturing and non-manufacturing sector in our model.

to calculate the share because the U.S. trade online database does not cover the earlier part of our sample.

ii. Trade between U.S. states

(a) Manufacturing: The CFS is available for years 2002, 2007, 2012, and 2017. We linearly interpolate the trade shares calculated from the CFS for all years between 2002 and 2016.[17] We multiply these shares by the level of manufacturing-sector domestic sales in the United States according to the WIOD for each year to obtain the level of trade flows between U.S. states.

(b) Non-manufacturing: The non-manufacturing sector includes both agriculture and services, so we assume that it is tradable. We estimate the elasticities from the gravity equation, using the CFS to impute the trade flows as in Section B.2.2.[18]

Measuring productivity catch-up. To estimate China's productivity catch-up process, we measure the productivity of the United States and China from the data. In our model, the productivity parameter of country n in sector j, z_t^{nj} is

$$\lambda_t^{nj,nj} = z_t^{nj} \left(\frac{w_t^{nj}}{P_t^{nj}} \right)^{-\theta \cdot \gamma^{nj}} .$$

We use this equation to measure z_t^{nj}. The construction of γ^{nj} is as described before. We construct the trade share sequence $\lambda_t^{nj,nj}$ from the EORA database because the WIOD does not cover the earlier part of the sample.[19] For wages and prices, we use Penn World Table (Version 9.1). We measure w_t^{nj} as the ratio between total labor compensation and total number of employment (variable *emp* in PWT), where labor compensation is calculated as the product of labor share (variable *labsh*) and GDP (variable *cgdpo*). We proxy for the index for manufacturing price using the average of the import and export prices of country n in period t (variables *pl_m* and *pl_x*, respectively):

B.3.3 Assessing the Importance of Evolving Beliefs

We assess the importance of accounting for evolving beliefs in terms of migration costs. The red dashed line in Figure B.1 plots the recovered perfect foresight path using the model with evolving beliefs. This path corresponds to the outcome that would have occurred if agents had perfect foresight in 2001. Compared to the actual allocation, shown by the blue solid line, the recovered perfect foresight path implies faster reallocation out of manufacturing. Intuitively, since agents anticipated the catch-up in China's manufacturing productivity, they moved out sooner.

Researchers who assume that the data are generated by agents with perfect foresight would view the blue solid line as the perfect foresight path. The difference between the blue solid line and the red dashed line therefore reflects the importance of accounting for evolving beliefs over 2001–2007 when explaining the change in manufacturing employment over this period. By 2007, when agents' beliefs became aligned with the actual process, the difference was around 0.2 percent points of employment or 0.3 million manufacturing employment.

To put these numbers in perspective, we consider the following experiment. We start from the red dashed path and conduct a counterfactual in which the cost of migrating from manufacturing

[17] For 2000 and 2001, we assume the trade shares are the same as in 2002.

[18] The estimates in equation (B.6) using the 2007 data are $\hat{\beta}_1 = 0.650$, $\hat{\beta}_2 = 0.776$, $\hat{\beta}_3 = -1.347$, and $\hat{\beta}_4 = 1.376$.

[19] Our quantitative analysis focuses on after 2000, which WIOD covers, however, we also show the evolution of the relative productivity between the U.S. and China since the early 1990s, which WIOD does not cover.

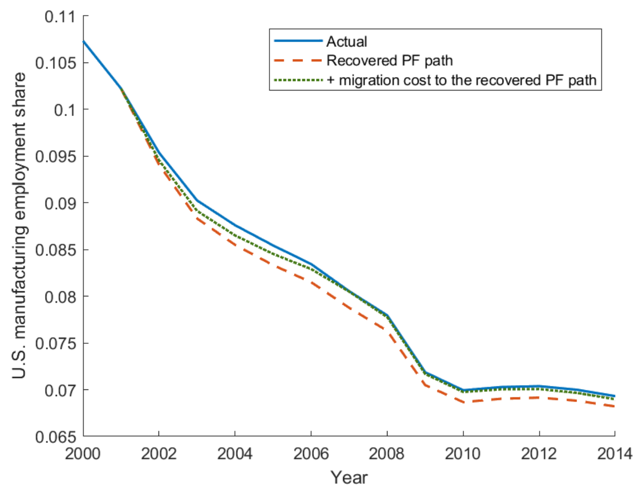

Figure B.1: Manufacturing Employment Share Under Higher Migration Cost

Note: This figure presents the actual manufacturing employment share (blue solid line); the path if agents would have had perfect foresight in 2001 (red dashed line), which is recovered from the actual evolution and the model with evolving beliefs; and the counterfactual path (green dotted line) obtained from adding 50% to the cost of migrating out of manufacturing over 2001–2007 on the recovered perfect foresight path (red dashed line).

to non-manufacturing is $(1 + x)$ times the original level over 2001–2007, the period over which agents' beliefs gradually converged to the true process. We then choose x so that this counterfactual path has the same manufacturing employment share as the data in 2007. Intuitively, we close the gap between the inferred perfect foresight paths using the model with evolving beliefs and using the perfect foresight model (which is the data) by introducing a higher cost of moving out of manufacturing to the former. The value of x gives us a sense of how important evolving beliefs are in terms of migration costs. We find that when $x = 50\%$, this counterfactual path (the green dotted line) has the same manufacturing share in 2007. Therefore, to account for the observed path of manufacturing employment share, the model with evolving beliefs needs approximately one-third lower costs of switching out from the manufacturing sector than the model with perfect foresight.