Grafting*

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Preliminary and Incomplete

May 30, 2026

Abstract

Using the universe of U.S. employer establishments from 1978 to 2022, we study grafting in the market for plants. Grafting occurs when firms buy existing plants in an industry rather than building them de novo. We document several new facts about this market. Grafting is quantitatively important, particularly for large and multi-plant firms. A substantial share of grafting occurs via partial acquisitions in which the buyer selects specific plants. Grafted plants are larger, more likely to survive than de novo plants, and grow faster post-acquisition. We find that grafting exhibits strong positive assortative matching in employment sizes across firms and between plants and acquiring firms. To interpret these patterns, we develop a tractable dynamic equilibrium theory of this market for selected plants, in which firms invest in plant creation and also search for and select acquisition targets. The theory accounts well for the characteristics of this market. After quantifying the model, we use it to study the aggregate implications of plant selection across firms and the impact of demographic changes on the supply of new plants.

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1 Introduction

The proverbial Main Street is in constant flux. Not because of new construction or big projects, we all know those happen rarely, but because its establishments change firm identity frequently. The corner that today hosts a McDonald's or a Citibank might next year host a Starbucks or a Bank of America. Firms, and particularly large firms, actively buy and sell establishments in a market for plants.[1] A common way for firms to add establishments in their industry is by 'grafting' them from other firms, rather than by building them themselves. What are the characteristics of this market for plants? What are the advantages for firms of grafting rather than building 'de novo'? What are the aggregate consequences of an active and well-functioning market for plants? In this paper, we use U.S. Census data on the universe of employer firms and plants from 1978–2022 to study the empirical characteristics of this market, propose a theory to understand it, and use it to draw aggregate implications of its role in the economy.

For the U.S. economy in 2019, more than xx% of all plants had changed firm ownership at least once in their lifetime. This share increases substantially for plants owned by multi-establishments firms, and increases fast with employment size. Thus, trading plants is common, and so the market for plants is large and active. Of course, there is also the well-studied market for firms, but between 1979 and 2022, about a quarter of grafting events were partial, and in those, on average, less than half of the plants were transacted. Partial grafting events differ from mergers and acquisitions because plants are very heterogeneous, so partial grafting involves establishment selection. If we look at plants owned by multi-establishment firms operating in a specific industry (i.e., 6-digit NAICS), the average employment size is about xx, but the two-standard deviation interval goes from xx to xx employees. Selecting the specific plants that firms want to graft or offload, therefore, matters. We show that grafted plants are larger across all plants and within the acquiring firm, survive longer than de novo plants, and grow faster than other plants from the acquiring firm.

The literature has already documented that, in the last half century, large firms have grown primarily by adding additional plants in more locations (Cao et al. (2017), Rossi-Hansberg et al. (2021), and Hsieh and Rossi-Hansberg (2023)). This growth has happened through grafting. We show that the share of grafted plants among large firms with more than 2500 employees is disproportionately high, at about one-third. So large firms use this market intensively to buy specific plants. This selected set of plants is mostly in the hands of other large firms, since we find evidence of positive assortative matching in the size of trading firms. Furthermore, larger buyers tend to buy larger plants among the ones owned by sellers. Namely, there is also positive assortative matching between the relative size of the grafted plants among those owned by the seller and the grafting firm's size relative to the seller's. Together, this evidence suggests that plants are born with (e.g., location) or develop (e.g., installed capital or customer base) specific characteristics that are matched differentially well with firms. The market of plants allows firms to select and graft the ones that are particularly well-suited for them. In this sense, the market for plants is a market of selection.

Armed with these facts, we propose a theory of firm dynamics in which firms create de novo establish-

[1] Throughout, we use the term establishment and plant interchangeably.

ments but also buy and sell plants in a market for plants. Plants use a decreasing returns to scale technology, with an effective productivity that depends on the plant's type, the firm's type, as well as a match-specific productivity that evolves stochastically. They face a fixed operating cost that is increasing in their type, and that will drive them to exit if their effective productivity is low enough. Given the observed assortative matching between firms and plants, we assume that productivity is log-supermodular in firm and plant type. Firms can create plants by paying a cost that is convex in their Poisson arrival rate. Alternatively, they can search in the market for plants by paying a cost that is also convex in the Poisson arrival rate of matches. The assumed convexity implies that growing firms will simultaneously create de novo plants and search and graft in the plant market. Firms that search in the market are randomly matched with other plant owners (potential sellers). Upon a match, the buyer draws a firm-plant specific productivity, and if there is surplus from a transaction, the parties divide it according to Nash bargaining. Hence, searching yields new draws of match-specific plant productivities that have option value. In fact, in equilibrium, the distribution of effective productivities of newly grafted plants will first-order stochastically dominate that of newly created plants.

The resulting model yields firms that produce using many plants. The boundary of the firm is simply determined by the firm type and the convex cost of either way of adding plants. Given a set of plants, a firm's profits are simply the sum of the profits of each plant. Our model does not include any other source of decreasing returns at the firm level, either technological, through span-of-control costs, or through cannibalization across plants (as in Oberfield et al. (2024)). We also abstract from market power or firm-level financial constraints. The linearity in plant profits provides the tractability to solve and characterize a model in which heterogeneous firms own many heterogeneous plants, which they partially graft by selecting plants from other heterogeneous firms with their own collections of heterogeneous plants. It implies that the evolution of plants depends solely on the firm that owns them through the resulting effective productivity of the plant. Since the evolution of the firm-plant specific component of productivity does not depend on the firm's actions, the firm's type is fixed, and plants are found randomly by searching buyers, the plant's type and plant productivity are all that is needed to solve for the plant dynamics and the equilibrium of the economy. We can then compute the characteristics of firms by simulating the model and the resulting matches.

We characterize the steady state of a version of the model in which the match-specific multiplicative productivity component of de novo or grafted establishments is drawn from a Pareto distribution. We study a specific limit case in which the lower bound of the distribution goes to zero, and the arrival rate of de novo plants or matches to potential graft opportunities goes to infinity. This limit guarantees that the arrival rate of plants with match-specific productivity above some level is only a function of the investment in new plants, or the search effort, and not the firm's productivity. Then, log-supermodularity in productivity between plant and firm type implies that high-type firms will create disproportionately more high-type plants, although the effective productivity distribution of these plants will be identical to that of other firms. The same result applies to grafted plants: high-type firms will graft more high-type plants, but the distribution of their effective productivity will be independent of firm type. Since more high-type plants are in the hands of high-type firms,[2] these results imply selection that yields assortative matching across buyers and sellers and across firms and plants. Upon a transaction, the productivity increase of the grafted plant will be distributed Pareto, and its expectation is given by the inverse of the Pareto coefficient of the match-specific productivity distribution.

The model has other realistic properties that can be characterized under more specific assumptions or numerically. For example, under specific functional form assumptions, we can characterize the upper tail of the distribution of effective productivities for a given plan type, or for de novo or all plants. In fact, the model can rationalize why, as we find in the data, the tail parameter of the distribution of effective productivity of all plants is the same as that of de novo plants. The reason is that, if the tail of the distribution of match-specific productivity is thin enough not to dominate, the tails of both distributions of plants are governed by the parameters that determine the availability of very good plants and firms, as well as positive sorting. We can use these results to estimate several model parameters.

We then proceed to use the facts we uncovered to quantify the model and explore its aggregate implications. In the first exercise, we study the aggregate implication of the market of selection. We compare an economy with our assumed log-supermodular technology between plants and firms, with a modular one. The modular economy does not yield any positive sorting, so average plant employment is smaller, and there is less grafting of high-type plants. We find that output per worker in the modular economy without grafting declines only marginally, but as much as 2% when we allow grafting. The result stems mostly from the effect grafting has on de novo plant formation. In a second application, we use the model to study the aggregate implications of the well-known decline in firm and establishment entry rates since the late 80's.[3] We document that the share of de novo plants created by entering firms, although large, declined from 90% to 82% during this period. We also document a declining share of grafted plants, particularly since the late 90's. We argue that declines in population growth can be linked to fewer de novo firms and, therefore, fewer de novo plants, which, in turn, result in fewer targets for grafting, slower grafting rates, and a decline in output per worker. The results of slowing down the market of plant selection.[4]

1.1 Related literature

The literature on plant dynamics and the market for plants is scattered across fields and topics. Rossi-Hansberg and Wright (2007) study establishment dynamics and their size distribution, but do not study their allocation across firms or the market for plants. Cao et al. (2017) documents that firms in the US grow by adding plants, and develops a theory where firms create plants to grow. (Rubinton, 2020; Aghion et al., 2023; De Ridder, 2024) also argue that the IT revolution has been important in the expansion of firms through plant creation. All these papers also abstract from the market for plants and the resulting

[2]Namely, the measure of plants of a given type operated by a specific type of firms is log-supermodular.

[3]As in Haltiwanger et al. (2012), Karahan et al. (2024), Engbom (2020), Rubinton (2020), among others

[4]This application is, for now, not included in this draft.

selection. Sadun et al. (2025) studies the role of management practices for the productivity of firms and their plants. Better-managed firms grow by creating better plants and improving acquired plants due to their better management practices. The finance literature has also studied the market for plants. Guntin and Kochen (2024) shows that the market for firms alleviates inefficiencies that stem from financial frictions. Rhodes-Kropf et al. (2005) and Rhodes-Kropf and Robinson (2008) present evidence of positive sorting between the acquirer and the acquiree, and provide a theory rationalizing the observation. Maksimovic and Phillips (2001) shows that there are efficiency gains resulting from the trade of plants. Braguinsky et al. (2015) obtained the same conclusion for the Japanese cotton industry. Relative to this literature, we provide a general equilibrium model of the market for plants, and its implications for the aggregate economy, and contrast it with data for the entire U.S. economy.

More broadly, the importance of capital reallocation across firms has been well recognized in the literature. Eisfeldt and Rampini (2006) documents substantial capital reallocation among firms. Hsieh and Klenow (2009) documents misallocation of capital across plants in the cross-section of establishments in China and India. Since then, a myriad of papers have examined the relevance of capital misallocation to various phenomena. Relative to them, we study more specifically the market for plants, its selected nature, and the frictions created by the need to search for trading partners in this market.

Of course, a large and related literature studies firm dynamics, the market for mergers and acquisitions, and their resulting implications. This literature, in general, does not consider the role of plants. Seminal papers in this literature include Luttmer (2011), which builds a theory of firm growth based on the creation of 'blueprints', and Klette and Kortum (2004), which studies a model where firms grow through the accumulation of new products and products that they compete away from others. A seminal paper in the market for firms is Jovanovic and Rousseau (2002), which argues that, as with investment, M&A activity is explained by a firms' market value relative to its replacement cost of capital, a firms' Q. A recent literature builds on this and quantitatively studies the effect of M&A activity for aggregate activity (David, 2021; Cavenaile et al., 2021; Bhandari et al., 2024; Hasenzagl, 2024; Milosavljević, 2025; Olmstead-Rumsey et al., 2025). This literature only studies full M&As and not partial grafting. hence selection of specific plants plays no role.

Firm expansion through the addition of plants is also important in studying spatial growth. Rossi-Hansberg et al. (2021) shows that the expansion of the number of plants of large firms across regions has led to increasing national but decreasing local concentration in product and service markets. Hsieh and Rossi-Hansberg (2023) argue that these large firms have added plants across regions by exploiting technologies with higher fixed and lower marginal costs. Oberfield et al. (2024) and Oberfield et al. (2024) studied the way plants sort across space within the US, in models where the boundary of the firm is limited by span of control costs. Relative to these papers, we abstract from the spatial dimension, but study the dynamics of the market for plants, which does not feature in any of them.

The international trade literature has also studied the expansion of firms through the opening of subsidiaries across countries to serve specific markets. Most of this literature is static and does not study the market for plants but only 'greenfield' investments. An exception is Nocke and Yeaple (2007) and Nocke and Yeaple (2008), which allow for cross-country mergers and acquisitions in a setting with complementarities between potential acquirers and target firms. Garetto et al. (2019) does not consider the market for firms or plants, but does study the multinational's dynamic problem of selecting in which countries to open subsidiaries.

Finally, we relate to the literature studying the decline in business dynamism (Davis et al., 2007; Haltiwanger et al., 2012) and its population origins (Alon et al., 2018; Bornstein et al., 2018; Engbom et al., 2019; Engbom, 2020; Hopenhayn et al., 2022; Peters and Walsh, 2022; Karahan et al., 2024). Related to this literature, we show that changes in population dynamics affect the creation of new plants. Then, we use the secular decline in population growth in the US to explain the changing dynamic patterns in plant creation and grafting across firms, and argue that it has important aggregate consequences in terms of productivity and welfare.

The rest of the paper is organized as follows. Section 2 presents details of our dataset and the main facts about grafting and the market for plants. Section 3 presents our theory and derives its main implications. Section 4 quantifies the theory and uses it to study the role of selection in the market for plants and the impact of changes in population growth. Section 5 concludes. An Appendix contains additional empirical facts, model details, computational algorithms, and further quantification details and results.

2 The Market for Plants in the United States

We start by documenting the basic characteristics of the U.S. market for plants. We use the Census Bureau's Longitudinal Business Database (LBD) for most of our empirical analysis. Our dataset includes the near-universe of employer firms in the US economy from 1978–2022. Specifically, it includes all firms that issued at least one W-2 form in the non-farm, private economy.[5] Importantly, it links each firm to the set of establishments in which it has a majority stake. Throughout, we refer to a 'firm' as the group of establishments in a 6-digit NAICS industry owned by a unique firm. Of course, a firm in the data might own establishments in different industries, in which case we break it down into distinct firms.

The Census defines an establishment as "a single, distinct physical location where business is conducted, services are rendered, or industrial operations are performed" (see Chow et al., 2021).[6] The LBD contains longitudinally-consistent establishment identifiers.[7] For each establishment-year pair, the LBD provides

[5] Firm-level identifiers in the LBD are known not to be longitudinally consistent. To address this, we use the methodology from Dent et al. (2018) to create longitudinally consistent firm identifiers.

[6] Note that establishment-level identifiers are not always tied to a physical location. To illustrate this, consider a store that operates at some location in year t but closes down during the same year. If a new store (with a different owner) becomes active in the same location in year $t + 1$, then the old and new store will be assigned *different* establishment-level identifiers.

[7] These identifiers are carefully constructed through an array of data sources from the Census Bureau, IRS, BLS, and SSA. The main idea behind these identifiers is whether an establishment continues operating with employment. In Appendix A.1, we describe in detail how the Census Bureau constructs these establishment-level identifiers.

information on employment,[8] payroll, 6-digit NAICS industry,[9] legal form of organization, and location (up to its zip code).

We also use data from the Economic Censuses for some exercises. Its main advantage is that it contains establishment-level revenues, and can easily be merged with the LBD.[10] However, these data are only available every 5 years (those ending in years 2 and 7).

2.1 The Market for Plants is Large and Active

An establishment is 'grafted' in year t if it is active, has the same identifier and industry in both years, but the firm f identifier changes from $t-1$ to t. Consider a firm f in year t that owns and operates P_{ft} plants. The mechanical evolution of the number of plants of this firm is given by

$$P_{ft} - P_{ft-1} = N_{ft} + G_{ft} - E_{ft} - O_{ft} \ . \tag{1}$$

where in year t, N_{ft} denotes the plants the firm creates de novo, G_{ft} denotes the plants it grafts from other firms, E_{ft}, denotes the plants that exit, and O_{ft} denotes the plants grafted by others (or the offloaded plants). Naturally, if we sum over all firms, the total number of grafted plants has to equal the total number of offloaded plants, namely, $\sum_f (G_{ft} - O_{ft}) = 0$.

Define the grafting rate g_{ft} as the total number of grafted plants relative to all plants.[11] It measures the rate at which active plants change ownership in the market for plants. Table 1 presents average, standard deviation, minimum, and maximum grafting rates in our data across years. We present these statistics for the whole economy and by major industry. Most of the firms in the U.S. economy have a single plant, which lowers the average grafting rate. Thus, we also present all statistics conditioning for plants owned by multi-plant firms.

Grafting rates are non-trivial for the economy as a whole and in all sectors. Among all plants, the average yearly grafting rate is 1.3%. Grafting rates for plants owned by multi-establishment firms are about twice as large. Grafting is pervasive across all major industries, especially among those owned by multi-plant firms. For these plants, Manufacturing (Manuf) has the highest average grafting rate at x.x%, while Education and Health Services (EU) has the lowest rate at x.x%. The literature has reported related and consistent rates of reallocation of plants and capital across firms.[12]

The grafting rate measures the frequency of grafting in a given year. Of course, most plants are only

[8] The number of employees in the payroll period that includes the March 12 calendar day. Employment is defined as the number of employees, regardless of full- or part-time status.

[9] Industry codes are made consistent over time using the methodology by Fort and Klimek (2018).

[10] The available Economic Censuses are the Census of Construction, Census of Manufactures, Census of Wholesale Trade, Census of Retail Trade, and Census of Services.

[11] That is, $g_{ft} = \sum_f G_{ft} / \sum_f P_{ft}$.

[12] Andrade et al. (2001) documents that around 3% of publicly listed firms are acquired each year. More recently, David (2021) reports an annual firm-acquisition rate of 3.7%. Maksimovic and Phillips (2001) documents that 3.9% of manufacturing plants change hands each year. Finally, Eisfeldt and Rampini (2006) measures, using Compustat data, yearly capital reallocation rates between 1.4% and 5.5%.

Table 1: Grafting rates, 1979-2022

		Average	Std. dev.	Min	Max
	Grafting rate, all plants	0.013	0.004	0.006	0.021
	Grafting rate, plants of multi-plant firms	0.025	0.008	0.009	0.041
MUC	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
Manuf	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
Trade	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
Transp	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
FIRE	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
Serv	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
EH	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx
EEA	Grafting rate, all plants	x.xxx	x.xxx	x.xxx	x.xxx
	Grafting rate, plants of multi-plant firms	x.xxx	x.xxx	x.xxx	x.xxx

Notes. MUC: Mining, Utilities, and Construction. Manuf: Manufacturing. Trade: Wholesale and Retail trade. Transp: Transportation and Warehousing. FIRE: Finance, Information, and Insurance. Serv: Professional, Management, and Administrative Services. EH: Education and Health Services. EFA: Entertainment, Food and Accommodation Services.

traded infrequently. Hence, an alternative way to gauge the level of activity in the market for plants is to compute, in the cross-section of plants in a given year, how many times each plant has been traded in its lifetime and report the resulting trade frequencies. Table 2 presents these results for the 2019 cross-section of plants. The rows of the table present the share of plants that have been grafted more than a particular number of times. Among the whole cross-section of plants, xx% have been sold at least once in their lifetime. Among plants owned by multi-plant firms, the share increases to xx%. Larger plants are substantially more likely to be traded. For example, conditioning on plants with at least 100 employees, the share rises to xx% and among those with 500 employees or more to xx.xx%. Naturally, these shares decline rapidly when we condition on multiple grafts. Still, among plants with more than 500 employees, more than x% of plants have been traded more than 3 times, a perhaps surprisingly large number.

Of course, many grafting events are simply acquisitions, in which a firm buys all the plants of another firm. Mergers and acquisitions across firms have been well studied in the literature. The market for plants, however, includes many instances in which only a fraction of the plants of the offloading firm are traded. In those cases, the offloading firm does not disappear or is fully absorbed by the acquiring firm, but continues

Table 2: Trade shares in the cross-sections of 2019 plants

	All plants	Plants owned by multi-plant firms	Plant empl.> 20	Plant empl.> 100	Plant empl.> 500
	(1)	(2)	(3)	(4)	(5)
Grafted ≥ 1	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
Grafted ≥ 2	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
Grafted ≥ 3	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
Grafted ≥ 4	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
Grafted ≥ 5	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
Grafted ≥ 6	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx

Notes. The table presents the frequency of grafting events for a given plant for the 2019 cross-section. Column (1) presents results for the whole cross-section of plants. Column (2) restricts to plants owned by multi-plant firms. Column (3) restricts to plants with at least 20 employees, column (4) restricts to plants with at least 100 employees, and column (5) restricts to plants with at least 500 employees.

operating with the remaining plants. These partial acquisitions of a fraction of the plants of a firm are different in nature, because a particular set of the plants of the offloading firm is traded. These plants are selected. Namely, they differ in specific ways from the plants that remain at the offloading firm, as we underscore below. These partial acquisitions make the market for plants distinct from the market for firms. In Table 3, we condition on grafting events by multi-plant firms and present results on the relevance of partial acquisitions. On average, across the grafting events of multi-plant firms, around 24% involve a partial acquisition. Moreover, on average, 41% of the plants acquired were grafted in partial trading events, with this average rising to 64% in specific years. Hence, even though the market for plants has received much less attention in the literature, its use seems as prevalent and important as the market for firms.

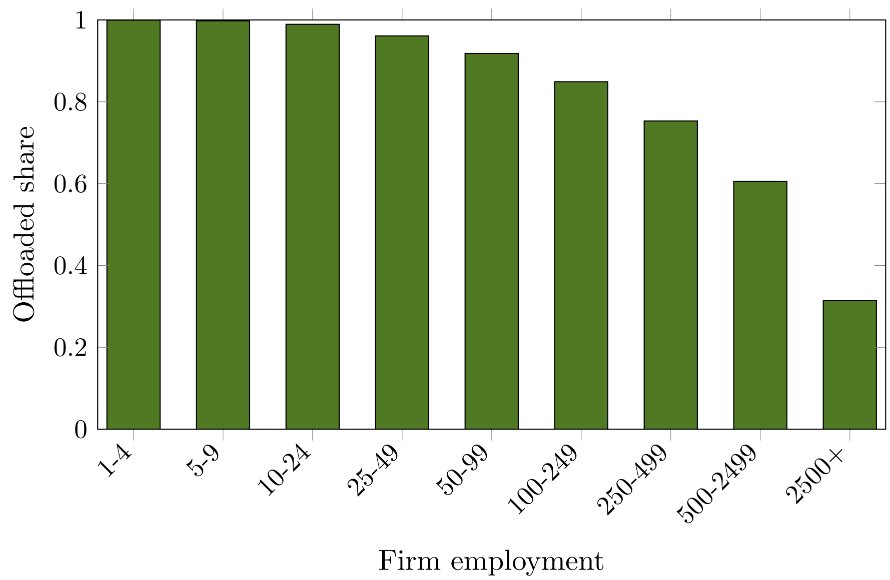
Table 3: Partial grafting events of multi-plant firms, 1979-2022

	Average	Std. dev.	Min	Max
Share of plants grafted in partial grafting events	0.412	0.147	0.112	0.636
Share of partial acquisitions	0.243	0.078	0.082	0.379

Notes. We condition on grafting events of multi-plant firms. The first row presents the share of plants grafted in year t in partial grafting events relative to all grafting events in that year. The second row presents the share of partial grafting acquisitions in year t relative to all acquisitions that year.

Not all firms are similarly active in the market for plants. Large firms are much more likely to offload only a fraction of their plants, and the share of plants that they acquire through grafting is larger, too. To substantiate these claims, we first present evidence on the share of offloaded plants in a grafting event by firm-size bin, controlling for industry-year fixed effects in Figure 1. For small firms, an offloading event typically involves the firm giving up all of its plants. As the employment size of firms increases, the average share of offloaded plants declines, and becomes as low as 30% for the largest firms. Hence, grafting events for small firms involve, in most cases, total acquisitions in which the offloading firm exits, while for the largest firms, partial acquisitions are the norm. In the latter case, the offloading firm continues to operate the remaining establishments.

Figure 1: Fraction of plants offloaded



Notes. The plot presents the average offloading share by size bin of the selling firm, after controlling for industry-year fixed effects. 2,098,000 observations for years 1979-2022.

We can also analyze scale dependence on the other side of the market, namely the characteristics of the firms that buy or graft the plants. An implication of an active market for plants is that firms build their plant portfolio by buying many of the plants that they operate. They can also create plants de novo. We compute the 'grafted' share of the firm as the ratio of the plants a firm acquired by buying them on the market relative to all the firm's plants. As before, we control for industry-year fixed effects, and now relate it to the grafting firm's employment. Figure 2 presents the grafted share by grafting firm total employment size for the year 2019.[13]

The grafted share is between 7 and 13% for the smallest firms, and grows to about a one-third for the largest buyers. Clearly, large firms obtain a considerable fraction of their plants by buying them in the market. However, firms of all sizes participate in the market for plants.

Table 4 reports grafted shares by major industry. Clearly, the scale dependence noted in for all firms in Figure 1 is also present in each sector. There is significant heterogeneity across sectors in grafted shares, but we note again that grafting is used by firms in all sectors to build their portfolio of plants.

[13]We do not observe if a plant was grafted prior to 1979, which results in under-representing the grafting rate in the initial years of the sample. Hence, we present here the results for 2019, the latest year available in the data that is not affected by the Covid pandemic.

Figure 2: Fraction of grafted plants

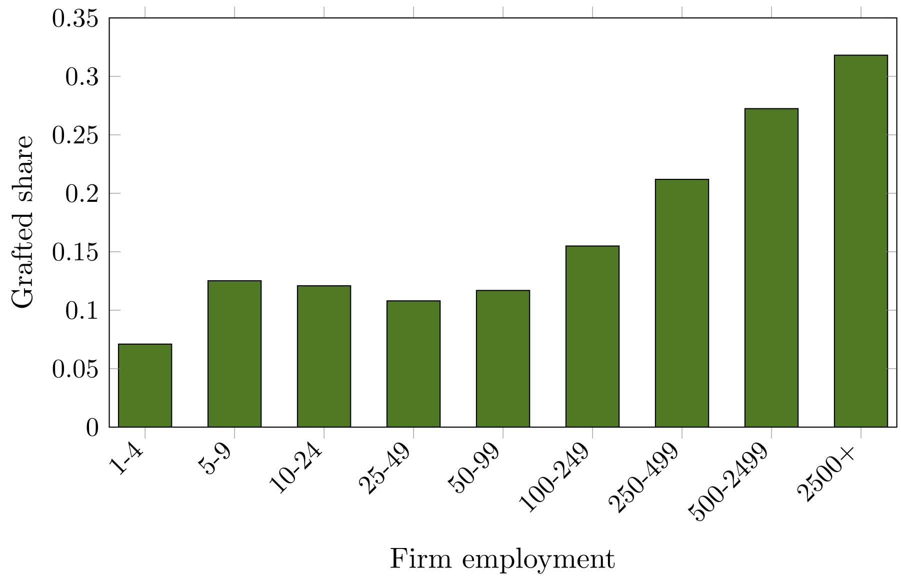


Table 4: Grafted share by industry

	MUC	Manuf	Trade	Transp	FIRE	Serv	EH	EFA
	Grafted share							
1-4 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
5-9 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
10-24 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
25-49 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
50-99 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
100-249 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
250-499 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
500-2499 empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx
2500+ empl.	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx	x.xxx

Notes. Grafted share is the share of outstanding plants of a firm that was grafted. We perform these calculations for the 2019 cross-section of firms and plants. MUC: Mining, Utilities, and Construction. Manuf: Manufacturing. Trade: Wholesale and Retail trade. Transp: Transportation and Warehousing. FIRE: Finance, Information, and Insurance. Serv: Professional, Management, and Administrative Services. EH: Education and Health Services. EFA: Entertainment, Food and Accommodation Services.

2.2 Plants within firms are different

Plants within an industry, and even within firm are very different. So, the plants sold in the market for plants are heterogeneous production facilities which match differentially well with specific firms. One source of heterogeneity across establishments is their industry. But establishments also differ significantly in employment size even within firms, and therefore, within industries.

Table 5 presents the mean of the average ln employment across plants, as well as the mean, standard deviation, minimum, and maximum of the standard deviation of ln employment across plants within firms. The minimum is always zero, given that there are always some firms with two identical plants in terms of employment, so the firm exhibits no dispersion across plants. In the whole economy and for all sectors, it is clear that the dispersion in employment across plants within firms is fairly large. For example, on average for the whole economy the 2 standard deviation interval includes plants with xx and xx employees. In manufacturing, the same interval covers plants with xx employees and plants that are x times larger.

Table 5: Plant dispersion within firms

	Dispersion within firm ln employment				
	Mean	Std. Dev.	Min	Max	Mean of average within-firm employment
All	x.xx	x.xx	x.xx	x.xx	x.xx
MUC	x.xx	x.xx	x.xx	x.xx	x.xx
Manuf	x.xx	x.xx	x.xx	x.xx	x.xx
Trade	x.xx	x.xx	x.xx	x.xx	x.xx
Transp	x.xx	x.xx	x.xx	x.xx	x.xx
FIRE	x.xx	x.xx	x.xx	x.xx	x.xx
Serv	x.xx	x.xx	x.xx	x.xx	x.xx
EH	x.xx	x.xx	x.xx	x.xx	x.xx
EFA	x.xx	x.xx	x.xx	x.xx	x.xx

2.3 Grafted plants are selected

As we showed above, firms hold a heterogeneous portfolio of plants. Here we show that when they sell some of them, they tend to sell a selected set. In particular, the plants that are grafted are on average larger than other plants. Figure 3 presents, for the cross-section of plants in 2019, the density of ln employment for grafted plants and plants that have never been grafted. Because plants belong to different industries, we normalize each ln plant size by subtracting the mean of ln employment in the plant's industry in 2019, and then dividing by the standard deviation of ln employment of plants within that industry in 2019. As seen in the figure, the distribution of grafted plants is shifted to the right, such that it first-order stochastically dominates the distribution of never grafted plants. For example, among plants owned by multi-plant firms, plants that have never been grafted have on average about xx employees, while plants that have never been grafted are about xx% larger.

We can be more precise in calculating the size premium associated with grafting. Let l_{pft} denote

Figure 3: Size distribution of plants, grafted and never grafted

[To be disclosed]

Notes. An observation is the ln of employment of a plant in 2019, standardized within industry. We normalize each observation by first subtracting the average of ln employment among all plants in the plant's industry in 2019, and then dividing by the standard deviation of ln plant's employment among all plants in the industry in 2019. The dotted line presents the density function of a standard normal distribution for comparison.

employment in plant p, owned by firms f, in year t. Denote also by $\mathbb{1}_{p=G}$ an indicator function equal to one if the plan was grafted in the past, and $\mathbb{1}_{p=EG}$ an indicator function equal to one if the plant is ever grafted, either in the past or in the future. We can then estimate the size premium, β_{lG} for grafted plant by estimating,

$$\ln l_{pft} = \alpha_l + \beta_{lG}\mathbb{1}_{p=G} + FE_{it} + \varepsilon_{pft}, \tag{2}$$

where FE_{it} is an industry-time fixed effect. We can also estimate the size premium by estimating the size of plants that have been, or will at some point in their life be grafted. Namely,

$$\ln l_{pft} = \alpha_l + \beta_{lEG}\mathbb{1}_{p=EG} + FE_{ft} + \varepsilon_{pft}, \tag{3}$$

where, in the most saturated version, FE_{fit} is a firm-year fixed effect. The results are presented in Table 6. Within an industry, plants grafted in the past or ever grafted are about 40% larger than those never grafted in the past or never grafted. Moreover, plants that are grafted at some point in their active life are also 28.5% larger within their current firm. In the last two columns of Table 6, we look at the selection of de novo plants that will be grafted in the future. These plants are also larger at inception: within an industry, relative to de novo plants that will not be grafted, they are 35.2% larger. Within firms, plants that will be grafted in the future are xx.xx% larger than the average de novo plant.

Grafted plants are not only larger but also less likely to exist. That is, grafted plants are selected not only on size, but also on their probability of survival. Let $\mathbb{1}_{Exit_{p\in t<t+5}}$ be a dummy equal to one if plant p exits in the next 5 years. Let $\mathbb{1}_{Gt}$ be a dummy indicating if the plant was grafted in year t and $\mathbb{1}_{Nt}$ an indicator function for de novo plants created in t. We estimate the following linear probability regression, which compares exit in the next 5 years of grafted or de novo plants in t with the rest of the population of plants. We also include an industry-year fixed effect.

$$\mathbb{1}_{Exit_{p\in t<t+5}} = \alpha_E + \beta_{ExG}\mathbb{1}_{Gt} + \beta_{ExN}\mathbb{1}_{Nt} + FE_{it} + \varepsilon_{pt}. \tag{4}$$

Table 7 presents the results of this regression. Unconditionally, the mean of $\mathbb{1}_{Exit_{p\in t<t+5}}$ is equal to 0.459. So almost half of all establishments fail in the next five years. Moreover, grafted plants are substantially less likely to fail than de novo ones. The 5-year failure probability of grafted plants is $\alpha_{Ex} + \beta_{ExG} = 0.464$, while that of de novo plants is $\alpha_{Ex} + \beta_{ExN} = 0.597$. The difference is as large as 13%. Thus, grafted plants

Table 6: Size premia for grafted plants

	Dependent variable: $\ln \ell_{efit}$				
	All	All	All	De novo	De novo
β_{lG}	x.xxx				
β_{lEG}		0.285***	0.410***	0.352***	x.xxx
		(0.023)	(0.018)	(0.019)	
α_l	x.xxx	2.277***	1.571***	1.083***	x.xxx
		(0.007)	(0.003)	(0.002)	
Industry-Year FE	✓		✓	✓	
Firm-Industry-Year FE		✓			✓
Observations	x	62,610,000	274,000,000	26,420,000	x

Notes. The table presents the results of the regression in (3). The dependent variable is the natural logarithm of a plant's employment. Coefficients relate to $\mathbb{1}_{p=G}$, which is a dummy equal to one if the plant was grafted in the past, and $\mathbb{1}_{p=EG}$ which is a dummy equal to one if the plant was ever grafted. Column (1) presents results looking at grafting in the past. Column (2) presents results for $\mathbb{1}_{p=EG}$ for all plants with Firm-Industry-Year fixed effects, while Column (3) uses Industry-Year fixed effects. Column (4) restricts attention to plants at the moment of their inception, and uses Industry-Year fixed effects, while column (5) uses Firm-Industry-Year fixed effects. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

are definitely selected based on their survival probability.

The results above show that grafted plants are larger and survive longer. Both results imply that grafted plants provide greater value to the firms that own them (and will naturally be more expensive as well). A simple back-of-the-envelope calculation, assuming that profits are proportional to plant size and that plants are kept until their exit, implies that grafted plants are about 80% more valuable for the firm.[14]

2.4 Positive assortative matching in grafting

The literature on mergers and acquisitions has provided evidence of positive assortative matching among firms that merge (Rhodes-Kropf et al., 2005; Rhodes-Kropf and Robinson, 2008; David, 2021). Some of this patterns are also present in the market for plants where we include partial acquisitions.

[14] Letting V_g denote the value of a grafted plant and V_{dn} the value of a de novo plant, in a stationary environment,

$$V_G = \frac{e^{\alpha_l + \beta_{lEG}}}{\alpha_{Ex} + \beta_{ExG}}, \text{ and } V_N = \frac{e^{\alpha_l}}{\alpha_{Ex} + \beta_{ExN}}.$$

Then, $V_G/V_N \approx 1.83$.

Table 7: Failure probability

	Dependent variable: $\mathbb{1}_{Exit_{p\in t<t+5}}$	
β_{ExG}	0.020***	0.041***
	(0.007)	(0.005)
β_{ExN}	0.153***	0.145***
	(0.004)	(0.003)
α_{Ex}	0.444***	0.444***
	(0.007)	(0.000)
Industry-Year FE		✓
Observations	274,000,000	274,000,000

Notes. The table presents the results of the regression in (4). The dependent variable is a dummy equal to one if the plant fails in the next 5 years. The coefficients are associated, respectively with $\mathbb{1}_{Gt}$ which is a dummy equal to one if the plant was grafted in year t and $\mathbb{1}_{Nt}$ which is a dummy equal to one if the plant is de novo in year t. Column (1) presents results with no fixed effect. Column 2 adds Industry-Year fixed effects. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

Consider a grafting event and let $\ln S_{bpt}$ denote the size of the buyer (or grafting) firm. Let $\ln S_{spt}$ denote the size of the seller (or grafted) firm. We consider to measures of size, either total firm ln employment or the average plant employment of the firm. We then estimate the following regression where β_S provides evidence of positive assortative matching.

$$\ln S_{bpt} = \alpha_S + \beta_S \ln S_{spt} + FE_{it} + \varepsilon_{pt}. \quad (5)$$

Table 8 presents the results. The first column uses the firm's total employment as the measure of size, while in second uses average plant employment as the size measure. For both measures we find significant evidence of positive firm-to-firm sorting ($\beta_S > 0$). Namely, large firms tend to graft plants from other large firms.

Firms not only graft plants from larger firms, but they are also more likely to graft their largest plants. Namely, in partial acquisitions, the buyer tends to prune the smallest plants of the seller. This pattern represents a form of positive sorting between grafting firms and grafted plants, which, we believe, is novel to the literature. Consider a grafting event and let S_{bpt}/S_{spt} denote the size of the seller relative to the buyer. Furthermore, let RS_{spt} denote the size of the grafted plant relative to the average size of the plants owned by the selling firm. We document sorting between firms and plants by estimating the following regression,

Table 8: Firm-to-firm sorting

	Dependent variable: $\ln S_{bpt}$	
	(1)	(2)
β_S	0.285***	0.230***
	(0.019)	(0.014)
α_S	−0.205***	−0.188***
	(0.043)	(0.028)
Industry-Year FE	✓	✓
Size measure:	Firm employment	Average plant employment
Observations	2,211,000	2,211,000

Notes. The table presents the results of the regression in (5). An observation is a grafting event. The dependent variable is $\ln S_{bpt}$, and the independent variable is $\ln S_{spt}$. The first column presents results where we use the firms' employment as the measure of size. The second column presents results where we use average plant employment as the measure of size. In both instances we use Industry-Year fixed effects. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

where we include industry-year fixed effects in the most saturated versions.

$$\frac{S_{bpt}}{S_{spt}} = \beta_P RS_{spt} + FE_{it} + \varepsilon_{pt}. \tag{6}$$

Table 9 presents the results. To measure the size of the firm and plant we use either employment or revenue. Revenue data is only available in the Economic Census, which includes only a subset of industries in the LBD. Moreover, since the Economic Census occurs every 5 years, we are restricted to comparing the size of firms and establishments only one year before each Census wave. Table 9 provides evidence of positive sorting between firms and plants. It shows that the larger the difference in size between the buyer and seller, the greater the difference between the grafted plant and the seller's average plant. Namely, large buyers select only the largest plants of smaller buyers.

2.5 Grafted plants grow faster

Before using these facts to motivate a theory of the market for plants, we present a final set of facts on the evolution of plants after they are grafted. When a plant is reallocated to a new firm, the plant becomes larger and its revenue increases. Let A_{pt} denote the age of the plant. We relate the change in log employment between period $t - dt_1$ and $t + dt_2$ to a grafting event happening at t, controlling for a plant's age and industry-year fixed effects. Thus, we estimate the following regression

$$\ln S_{pt+dt_2} - \ln S_{pt-dt_1} = \alpha_g + \beta_g \mathbb{1}_{Gt} + \beta_a A_{pt} + FE_{it} + \varepsilon_{pt}, \tag{7}$$

The coefficient of interest is β_g, which measures the effect of grafting on plant growth. The results are

Table 9: Plant-to-firm sorting

	Dependent variable: $\frac{S_{bpt}}{S_{spt}}$			
β_P	x.xxx	x.xxx	0.147***	0.177***
			(0.023)	(0.029)
Industry FE			✓	✓
Year FE			✓	✓
Industry-Year FE	✓	✓		
Average size of seller includes offloaded plants	✓		✓	✓
Size measure:	Employment	Employment	Employment	Revenue
Source	LBD	LBD	EC	EC
Observations	x	x	125,000	125,000

Notes. The table presents the results of the regression in (6). * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

presented in Table 10. The first column presents results on the change in ln employment between year $t-1$ to $t+1$ of a plant that is grafted in year t, relative to plants that did not change ownership in the LBD. Grafted plants grew x.x% throughout these two years, or x.x% per year. The remaining columns present results for different time windows before and after the grafting event, across different datasets and measures of size. Throughout, we find evidence that grafted plants grow faster than other plants, controlling for age, industry, and time period.[15] Hence, by reallocating plants across firms, grafting seems to increase firm size, probably by improving the match between firms and plants. Of course, transacted plants are not necessarily a conditionally random set, and so both selection and treatment can explain these results. We now proceed to develop a theory of the market for plants based on the facts uncovered in this section.

[15]These findings are consistent with Maksimovic and Phillips (2001), which documents efficiency gains when plants are acquired for a subsample of manufacturing firms.

Table 10: Grafting and growth

	Dependent variable: $\ln S_{pt+dt_2} - \ln S_{pt-dt_1}$					
β_g	x.xxx	x.xxx	x.xxx	0.030**	0.021**	0.053***
				(0.012)	(0.010)	(0.014)
α_g	x.xxx	x.xxx	x.xxx	0.141***	0.100***	0.129***
				(0.004)	(0.005)	(0.003)
Industry-Year FE	✓	✓	✓	✓	✓	✓
A_{pt} control	✓	✓	✓	✓	✓	✓
$\{dt_1, dt_2\}$	$1, 1$	$3, 3$	$3, 2$	$3, 2$	$3, 2$	$3, 2$
Size measure:	Employment	Employment	Employment	Employment	Revenue	Payroll per worker
Source	LBD	LBD	LBD	EC	EC	EC
Observations	x	x	x	21,590,000	21,590,000	21,590,000

Notes. The table presents the results of the regression in (7). * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

3 A Theory of the Market for Plants

The economy consists of households, firms, and plants that consume and produce, respectively, a single homogeneous good. We assume perfect competition in the goods market and normalize the price of goods to one. All households are identical, and population grows at a constant rate $\dot{L}_t/L_t = g_L$. Workers supply their unit of time inelastically in a perfectly competitive labor market at the equilibrium real wage, w. We will characterize only the steady state of this economy, so we eliminate time dependence from all notation.

All the model's richness lies in the problem of the firm and in the creation and exchange of plants. Firm entry requires Φ^E units of labor.[16] Upon entry, a firm draws a permanent type q from a distribution with

[16] We cannot allow for firm-wide fixed operating costs that might cause a firm to exit. If we did, an exiting firm would sell all its plant portfolio at once, an event we do not model.

cumulative density function (CDF) $F_q(\cdot)$. To produce and expand, firms need to create or buy plants. When a firm creates a plant, a process we will discuss in detail below, it draws a permanent plant type a from a distribution with CDF $F_a(\cdot)$, as well as a plant-firm-specific productivity z that evolves stochastically and is drawn from a distribution with CDF $F_z(\cdot)$ for newly created plants.

A plant of type a with match-specific productivity z owned by a firm of type q has effective productivity $X(q, a, z)$ and must pay a fixed operating cost of Φ_a units of labor. If a firm decides not to pay the operating cost for a plant, the plant shuts down permanently. Labor is the only input in production, and each plant's production exhibits decreasing returns to scale. Hence, the instantaneous profit of a plant of type a owned by a firm of type q, with match-specific productivity z, is given by

$$\max_l \frac{X(q,a,z)^{\alpha} l^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} - wl - w\Phi_a = w^{1-\frac{1}{\alpha}} X(q,a,z) - w\Phi_a \tag{8}$$

In what follows, we assume that $X(q, a, z)$ is increasing in all arguments and that it is log-supermodular in q and a. This assumption will allow the model to generate the type of selection we saw in the data. We summarize it in the following assumption:

Assumption 1 *The function $X(\cdot)$ can be expressed as*

$$X(q, z, a) \equiv h(q, a)z,$$

where $h(\cdot)$ is log-supermodular and increasing in both arguments.

We assume that each plant's match-specific productivity z evolves according to a geometric Brownian motion $d \log z = \mu dt + \sigma dW$. At this point, we make no assumption about the correlation of changes in z across a firm's plants.

3.1 Plant Creation and Grafting

At any point in time, firm f owns a finite (possibly empty) set of plants, $\mathcal{I}_f$. Each plant $p \in \mathcal{I}_f$ is characterized by its plant type a_p and its match productivity, z_{fp}.

A firm can incur a cost $c(n)$ in units of labor to try to create new plants. For a firm that pays $c(n)w$ to create plants, new plants arrive according to a Poisson process with rate $n\tilde{\Upsilon}$. $\tilde{\Upsilon} > 0$ is a constant that governs the flow of new plant entry. We assume that $c(\cdot)$ is increasing and convex in n. Upon the arrival of a new plant, the firm draws its permanent type and its initial match-specific productivity.

Firms can also search to graft plants in the market. Firms pay a cost $c(s)$ in labor units to search for other firms from whom to buy plants. When a firm searches, it encounters target firms according to a matching process. If a firm f encounters a potential target firm f', the potential buyer f draws a new, random match-specific productivity z_{fp} for each of the target plants $p \in \mathcal{I}_{f'}$. If the two firms mutually agree to transfer ownership of any subset of the target's plants, this can be done at no cost. If there are gains from trade, the two firms bargain over the price à la Nash, with buyer's bargaining power β.

If firm f exerts search effort $c(s_f)w$, target firms arrive according to a Poisson process with arrival rate $s_f\tilde{M}/S$ where $\tilde{M}$ is the flow of matches between buyer and seller firms and $S \equiv \int s_f df$ is the cumulative search intensities across all firms.

Each firm may also be found as a potential target. For an individual firm, this happens with arrival rate $\tilde{M}/J$, where J denotes the measure of firms.

In what follows, we assume that the initial match-specific productivities after a plant is created or a target found are drawn from a Pareto distribution. Further, we study a limiting economy in which the lower bound of the Pareto distribution is small and normalize the arrival rates so that the limiting economy is well-behaved. In particular, we assume:

Assumption 2 *The distribution of initial match-specific productivities, $F_z(s)$, and the arrival rates of new and target plants, $\tilde{\Upsilon}$ and $\tilde{M}$ satisfy:*

1. *$F_z(z) = \Pr(z_{fp} > z) = \underline{z}^{\zeta} z^{-\zeta}$, for $z \geq \underline{z}$, where ζ governs the tail, and $\underline{z}$ the lower bound, of the Pareto distribution.*
2. *If $\tilde{M} \equiv M\underline{z}^{-\zeta}$ for a matching intensity M, and $\tilde{\Upsilon} = \Upsilon\underline{z}^{-\zeta}$ for a new plant intensity Υ, we let $\underline{z} \to 0$.*

Together, these assumptions imply that when a plant exerts effort n and s, new plants with match-specific productivity greater than z arrive at rate $n\Upsilon z^{-\zeta}$ and potential target plants with whom the firm would have match productivity greater than z arrive at rate $s\frac{M}{S}z^{-\zeta}$.[17]

3.2 The Firm's Problem

We now describe the problem of a firm in a stationary economy. A firm's state is its type q, its set of plants, $\mathcal{I}_f$, and the state of each of those plants $\{a_p, z_{fp}\}_{p\in\mathcal{I}_f}$. We construct an equilibrium [18] in which the firm's value function can be expressed as

[17]The first part of the assumption–that match-specific productivities are Pareto-distributed–is relatively standard. The second part may require some explanation. When a firm encounters a plant (either new or existing), it draws a match-specific productivity and decides whether to purchase the plant. Firms with higher q are more likely to end up with an effective productivity $x = h(q, a)z$ that is high enough to keep/purchase the plant. Before taking the limit, $\underline{z}$ is strictly positive, and a firm with high enough q would keep purchasing *every* plant it encounters. As a result, having a higher q would have no impact on the purchasing decisions of these high q plants. In the limit, there are more encounters, but each encounter is likely to have a lower match-specific productivity. The limit is taken in such a way that the arrival rate of "good" matches stays the same, but there are extra encounters with low-productivity matches, most of which are discarded anyway. But the possibility of low-productivity matches means that higher q firms are more likely to purchase plants, even for very high q. To see that taking the limit simply adds many low productivity encounters to the economy, note that before taking the limit, when a plant exerts effort n and s, new plants with match-specific productivity greater than z arrive at rate $\Upsilon n z^{-\zeta}$ and potential target plants with whom the firm would have match productivity greater than z arrive at rate $s\frac{M}{S}z^{-\zeta}$, for $z > \underline{z}$. When we take the limit, the "for $z > \underline{z}$" is replaced by "for all $z > 0$". This type of limiting assumption has been used in Oberfield (2018), Buera and Oberfield (2020), and Borovičková and Shimer (2024), and has its roots in Houthakker (1955) and Kortum (1997).

[18]We do not have a proof of equilibrium uniqueness, but we conjecture that the value functions always have the structure in the equilibrium we construct.

$$\mathcal{V}\left(q,\{a_p,z_{fp}\}_{p\in I_f}\right)=w\left[\sum_{p\in I_j}v_{a_p}\left(h\left(q_f,a_p\right)z_{fp}\right)+u\left(q\right)\right]. \tag{9}$$

The first term denotes the value of the firm embedded in its collection of plants, where $v_a(x)$ denotes the value of a plant with permanent type a and effective productivity $x = h(q,a)z$. The Hamilton-Jacobi-Bellman (HJB) equation for the plant is given by

$$rv_a(x)=w^{-\frac{1}{\alpha}}x-\Phi_a+\frac{\tilde{M}}{J}\int_0^J\mathbb{E}_z\max\left\{0,\frac{P_a\left(zh\left(q_f,a\right),x\right)}{w}-v_a(x)\right\}\frac{s_f}{S}df+\mu xv_a'(x)+\frac{\sigma^2}{2}x^2v_a''(x)$$

.

The first term in this equation denotes the plant's instantaneous profits, the second, the fixed operating costs, the third the option value of the plant being found and sold to another firm that pays a price $P(\cdot)/w$ higher than its value for the current owner, and the forth and fifth account for the evolution of the value as the match-specific productivity evolves according to the assumed Browninan motion.

Given the plant value, we can easily determine the price of a plant. Recall that β denotes the buyer's bargaining power and that buyers and sellers bargain à la Nash. Hence, the price for a plant of type a where the seller's effective productivity is x_s and the buyer's effective productivity is realized to be x_b, satisfies

$$\begin{aligned}\frac{P_a\left(x_b,x_s\right)}{w} &= \arg\max_P\left[v_a\left(x_b\right)-\frac{P}{w}\right]^{\beta}\left[\frac{P}{w}-v_a\left(x_s\right)\right]^{1-\beta}\\ &= \beta v_a\left(x_s\right)+\left(1-\beta\right)v_a\left(x_b\right).\end{aligned}$$

The second term in the firm's value function is the value of the firm's potential for expansion, with HJB given by

$$\begin{aligned}ru(q) &= max_{n,s}\Bigg[n\tilde{\Upsilon}\mathbb{E}_{z,a}\left[v_a\left(zh(q,a)\right)\right]-c(n)\\ &\quad+\frac{s\tilde{M}}{S}\frac{1}{J}\int_0^J\sum_{p\in I_f}\mathbb{E}_z\left[\max\left\{0,v_{a_p}\left(h(q,a)z\right)-\frac{P_{a_p}\left(zh\left(q,a_p\right),z_{fp}h\left(q_f,a_p\right)\right)}{w}\right\}\right]df-c(s)\Bigg].\end{aligned}$$

The first and second terms denote the expected value from new plant creation minus its cost, while the third and fourth denote the expected value from grafting minus the search cost. We let $n(q)$ and $s(q)$ denote the optimal investment in de novo plants and, respectively, the optimal search effort, by firms of type q.

A plant of type a exits when its effective productivity x reaches a threshold $\underline{x}_a$, such that $v_a\left(\underline{x}_a\right)=0$. As usual, the optimal exit threshold satisfies the smooth pasting condition, $v_a'\left(\underline{x}_a\right)=0$. These conditions, along with the HJB and a no bubble condition, fully characterize the value function v_a. As the HJB and these boundary conditions make clear, a key feature of the model's formulation, which provides the tractability needed to solve it, is that one can follow individual plants without keeping track of the firm that owns them. Decisions about a plant depend on its firm only through x; given a and x, nothing else about the firm matters for the subsequent evolution of x or decisions about offloading or exit.

Free entry requires that the expected value of a firm upon entry but before its type realization and before building a portfolio of plants, is equal to the entry cost, Φ^E. That is,

$$\Phi^E = \int u(q) dF_q(q). \tag{10}$$

3.3 The distribution of plants

We next turn to the distribution of plants. The distribution of plants evolves due to the evolution of the match-specific productivity of the plant, de novo plants created by firms, and grafted plants that draw a new match-specific productivity and are owned by a different firm.

Let $\tilde{\eta}_a$ be the flow of de novo plants of type a, and let $\eta_a \equiv \tilde{\eta}_a / J$ be the flow normalized by the measure of firms. Under Assumption 2, η_a is given by

$$\begin{aligned} \eta_a &\equiv \Upsilon \int n(q) \int_0^\infty 1\left\{h(q,a)z \geq \underline{x}_a\right\} \zeta z^{-\zeta-1} dz dF_q(q) F_a'(a) \\ &= \Upsilon \underline{x}_a^{-\zeta} \int n(q) h(q,a)^\zeta dF_q(q) F_a'(a). \end{aligned}$$

Defined also b_a as the arrival rate of a potential grafter with match-productivity larger than $\underline{x}_a$, namely,

$$b_a \equiv \underline{x}_a^{-\zeta} \frac{M}{J} \frac{\int h(q,a)^\zeta s(q) dF_q(q)}{\int s(q) dF_q(q)}. \tag{11}$$

Then, the stationary measure of plants of a given type across effective productivities, $G_a(x)$, satisfies the Kolmogorov-forward equation (KFE) in Proposition 1. We relegate all proofs to the Appendix.

Proposition 1 *Among plants of type a, the stationary measure of plants per firm with effective productivity x, $G_a(x)$, satisfies the KFE*

$$\begin{aligned} 0 \;=\; & -g_L G_a'(x) - \frac{d}{dx}\left[\mu x G_a'(x)\right] + \frac{d^2}{dx^2}\left[\frac{\sigma^2}{2} x^2 G_a'(x)\right] + \underbrace{\eta_a \zeta x^{-\zeta-1} \underline{x}_a^\zeta}_{\textit{De novo plants that become } x,a} \\ & - \underbrace{G_a'(x)\, b_a (x/\underline{x}_a)^{-\zeta}}_{\textit{plants with } x,a \textit{ that are acquired}} + \underbrace{\zeta x^{-\zeta-1} \underline{x}_a^\zeta b_a G_a(x)}_{\textit{grafting of plants that become } x,a}, \end{aligned}$$

with initial and terminal conditions,

$$\begin{aligned} G_a\left(\underline{x}_a\right) &= 0, \\ G_a'\left(\underline{x}_a\right) &= 0, \\ \eta_a &= \frac{\sigma^2}{2}\underline{x}_a^2 G_a''(\underline{x}_a) + g_L G_a(\infty). \end{aligned}$$

The last boundary condition guarantees that the entry of new plants of type a is equal to the endogenous exit at $\underline{x}_a$, and the loss of mass at the upper tail as the mass shifts due to population growth.

3.4 Equilibrium

Consider first the equilibrium condition in the market for plants. For a firm of type q, the optimal choice of search, $s(q)$, satisfies the first-order condition

$$c'\left(s(q)\right) = \frac{M}{S}\beta \int h_(q,a)^{\zeta} \int \int_0^{\infty} \max\left\{0, v_a(\tilde{x}) - v_a(x)\right\} \zeta \tilde{x}^{-\zeta-1} d\tilde{x} dG_a(x) da \tag{12}$$

Integrating over q, and noting that the optimal search effort is a function of the average flow of matches relative to cumulative search effort, M/S, market tightness in the market for plants satisfies, namely,

$$\frac{S}{J} = \int s(q; M/S) dF_q(q). \tag{13}$$

If we specify the matching function to be constant returns in J and S, this equation alone determines plant market tightness in equilibrium, S/J. If we specify the matching function to be constant returns in J, S, and the population size L, then we need to use the labor-market-clearing condition below to determine S/J and J/L.

We now turn to labor market clearing. A plant with effective productivity x produces $y(x) \equiv \frac{1}{\alpha} w^{-\frac{1-\alpha}{\alpha}} x$ and uses $l(x) \equiv \frac{1-\alpha}{\alpha} w^{-\frac{1}{\alpha}} x$ units of labor for its variable cost. Market-clearing in the labor market is guaranteed if

$$L = J\left\{\int \left[c\left(n(q)\right) + c\left(s(q)\right)\right] dF_q(q) + \int\int \left[l(x) + \Phi_a\right] dG_a(x) da\right\} + \Phi^E \dot{J}. \tag{14}$$

The supply of labor is given by total population, while demand is given by the cost of investments in de novo plants and search, labor used in production, labor used in plant fixed operating costs, and firm entry costs. In steady state, new firm entry is only the result of population growth, and so $\dot{J} = g_L J$. So the labor market-clearing condition can be rearranged as

$$1 = \frac{J}{L}\left\{\int \left[c\left(n(q)\right) + c\left(s(q)\right)\right] dF_q(q) + \int\int \left[l(x) + \Phi_a\right] dG_a(x) da + \Phi^E g_L\right\}, \tag{15}$$

which determines the measure of firms per worker, J/L. We can then obtain the wage, which is equal to output per worker, by integrating across plant output, namely,

$$w = \frac{Y}{L} = \frac{J}{L} \int \int y(x) dG_a(x) da. \tag{16}$$

The above conditions together construct a competitive equilibrium of the model. More mechanically, we can solve for equilibrium with the following algorithm:

1. Outer loop: Guess the wage, w
2. Inner loop: Guess a schedule of $\{b_a\}$
   (a) Solve for the plant value function v_a, and exit threshold $\underline{x}_a$ for each a
   (b) Solve for de novo effort, $n(q)$
   (c) Solve for flow of new plants, η_a
   (d) Use KFE to solve for the measure of plants, G_a
   (e) Use the equilibrium conditions to solve for $s(q)$, J/S, and J/L
   (f) Update guess of b_a
3. Update guess of w

We now proceed to characterize the equilibrium that we have constructed.

3.5 Characterization

Consider first the creation and grafting of plants of a given type a. For a firm of type q, the arrival of de novo plants with type a and an effective productivity x, with $x \geq \underline{x}_a$, is given by

$$h(q,a)^{\zeta} n(q) \Upsilon \zeta x^{-\zeta-1} F'_a(a) \ . \tag{17}$$

In contrast, for the same type of firm q, the grafting rate of plants of type a with effective productivity x is given by

$$h(q,a)^{\zeta} s(q) M \zeta x^{-\zeta-1} G_a(x) \ . \tag{18}$$

The expressions in (17) and (18) are similar except that grafted plants are selected. This is reflected in the fact that the arrival of a de novo plant is proportional to $F'_a(a)$, the unconditional density that a plant is of type a, whereas the arrival rate of a grafted plant is proportional to $G_a(x)$, the measure of plants in the cross section of type a with productivity weakly less than x. $F'_a(a)$ and $G_a(x)$ are different in two respects. First, since plant survival may differ by plant type, the composition of plant types in the cross-section may differ from its denovo distribution, $F'_a(a)$. Second, when a firm creates a de novo plant of type a, it keeps the plant if its effective productivity exceeds the exit threshold, $x > \underline{x}_a$. In contrast, when a firm encounters a type-a plant with whom it can obtain effective productivity x, it acquires the plant only if x exceeds the potential seller's effective productivity. These two types of selection shape the differences between the type of de novo and grafted plants a firm q adds to its portfolio.

Clearly, in both cases, the arrival of type a plants is increasing in $h(q,a)$. Higher $h(q,a)$, combined with IID z realizations, implies that the effective productivities $x = h(q,a)z$ obtained by the match are higher too. Hence, a higher $h(q,a)$ delivers a higher arrival rate of potential matches with a given x. Since, by Assumption 1, $h(q,a)$ is log-supermodular, a high-q firm has disproportionally higher arrival rate of high-a de novo and grafted plants. We summarize the implications for the distributions of plant types and effective productivities in the following proposition.

Proposition 2 *The share of plants of type a created by firms of type q is given by*

$$\frac{h(q,a)^{\zeta} n(q) F'_q(q)}{\int h(\tilde{q},a)^{\zeta} n(\tilde{q}) F'_q(\tilde{q}) d\tilde{q}} \, , \tag{19}$$

and among plants of type a and productivity x that have never been grafted, the fraction owned by a firm with type q is also given by (19). Similarly, among the flow of grafted plants, the share of plants of type a acquired by firms of type q is given by

$$\frac{h(q,a)^{\zeta} s(q) F'_q(q)}{\int h(\tilde{q},a)^{\zeta} s(\tilde{q}) F'_q(\tilde{q}) d\tilde{q}} \, , \tag{20}$$

and among plants of type a that have been grafted at least once and that have productivity x, the fraction owned by a firm with type q is also given by (20).

Furthermore, among de novo plants of type a created by firms of type q, the distribution of effective productivity at plant birth is given by a Pareto distribution with density

$$\zeta x^{-\zeta-1} \underline{x}_a^{\zeta} \, . \tag{21}$$

Similarly, among plants of type a grafted by firms of type q, the distribution of effective productivity upon grafting is given by

$$\frac{\zeta x^{-\zeta-1} G_a(x)}{\int_{\underline{x}_a}^{\infty} \zeta \tilde{x}^{-\zeta-1} G_a(\tilde{x}) d\tilde{x}} \, . \tag{22}$$

Two comments about these results are in order. First, due to Proposition 1, the distributions in (19) and (20) are both log-supermodular. Which means that high-type plants are disproportionally created by high-type firms. Namely, there is positive sorting between firm and plant types.

Second, the distributions in (21) and (22) are independent of q. This means that for any given plant type a, plants created by high-q firms are no different from those created by low-q firms. The only difference in the composition of plant creation among firms is that high-q firms create more high-a plants. Similarly, plants of a given type a grafted by high-q firms are the same as those grafted by low-q firms, but high-q firms graft more high-a plants.

Since firms only graft plants if the resulting effective productivity is higher than the current effective productivity of the plant, the result that the distribution of plants grafted by a firm of type q is independent of the firm's type, implies that the productivity increment after a plant is grafted is Pareto distributed with shape ζ and lower bound given by the firm's previous productivity. This yields the following result.

Proposition 3 *If x_0 is the plant's productivity before a grafting event and x is the productivity after the event, the proportional increase in effective productivity is Pareto-distributed, with the density of x given by $\zeta x^{-\zeta} x_0^{\zeta}$, and the expected log-increase in effective productivity given by*

$$\mathbb{E}\left[\ln\left(x/x_0\right)\right] = 1/\zeta \ . \tag{23}$$

This result will be useful for estimating the parameter ζ, since we measured the average increase in output per worker following a grafting event in the data.

Before the realization of the match-specific productivity z, the expected profitability of a plant of type a for a firm of type q depends on the balance between its expected effective productivity, which is governed by $h(q, a)$, and its fixed operating costs, Φ_a. We now further characterize the equilibrium allocation for the case in which the increase in the expected productivity of a higher-type plant dominates the increase in its operating costs for all firm types. The assumption guarantees that higher-type plants are more desirable for all firms and, because of Assumption 1, particularly for high-q firms.

Assumption 3 *For all q, $h(q, a)/\Phi_a$ is increasing in a.*

Under Assumption 3, high-a plants are more profitable on average, and so newly created plants are disproportionately likely to be sufficiently productive to overcome the overhead cost and enter. Furthermore, higher-a plants tend to see a higher arrival rate of potential grafters whose effective productivity exceeds the exit threshold, namely, b_a is increasing in a. Finally, because potential grafters are more likely to operate a plant profitably the higher its type, high-a plants are also more likely to have been grafted in the past. We summarize these results in the following proposition.

Proposition 4 *Under Assumption 3,*

1. *the fraction of potential de novo plants of type a that are born with productivity high enough to enter, $\frac{\eta_a}{F_a'(a)}$,*

2. *the arrival rate of grafters with match productivity above $\underline{x}_a$, b_a,*

3. *the prevalence of plants of type a in the cross section relative to the prevalence of type a plants among potential de novo plants, $\frac{G_a(\infty)}{F_a'(a)}$,*

4. *and the fraction of plants of type a in the cross-section that have been grafted in the past,*

are all increasing in plant type a.

In order to characterize the distribution of plants across firms, we need to understand the rate at which plants of each type exit. In the Appendix, we show that the exit threshold, $\underline{x}_a$ can be expressed as

$$\underline{x}_a = \frac{w^{\frac{1}{\alpha}}\Phi_a}{\hat{H}(b_a)}, \tag{24}$$

where $\hat{H}(\cdot)$ is a strictly increasing function. The expression shows that there are two main motives for plant exit. First, when the fixed cost Φ_a is higher, plants are less profitable at any level of productivity, so they are more likely to exit in response to a declining match-specific productivity x. This is a standard force in Hopenhayn (1992). Second, a higher arrival rate of potential grafters, b_a, increases the option value of being acquired in the future, making it more attractive to survive, which lowers the exit threshold. This force has been emphasized in the context of firm mergers by David (2021). Which of these forces dominates depends on how quickly Φ_a rises with a, as well as on the equilibrium forces that determine b_a. We now further characterize the equilibrium under the assumption that Φ_a increases fast enough with a, such that the first force dominates and $\underline{x}_a$ rises with a.

Assumption 4 Φ_a *increases fast enough with* a *such that* $\underline{x}_a$ *is increasing in* a.

Assumption 4 implies that de novo high-a plants need to be more productive to survive. Hence, on average, they are larger (in terms of employment or output), more likely matched to high-q firms, and therefore more likely to have high effective productivity and survive. Furthermore, because, as we argued before, high-q firms create a larger share of de novo high-a plants, their de novo plants are larger and more likely to survive than those of low-q firms. Exactly as we saw in the data. We summarize these results in the following proposition.

Proposition 5 *Under Assumption 4,*

1. *high-a de novo plants are, on average, larger at birth and more likely to survive;*

2. *de novo plants of high-q firms are, on average, larger at birth and more likely to survive than de novo plants of low q firms.*

Together, these results imply that the plants available for grafting in the market for plants have already been selected. Furthermore, when firms graft a plant, the new effective productivity of the plant must be greater than its effective productivity at its old firm for the surplus of the trade to be positive. Therefore, when firms graft plants, they tend to operate at a larger scale and be more productive than before. Since targets are found randomly, the implication is that, for any firm, the size (measured either in employment or output) and productivity distributions of newly grafted plants stochastically dominate those of de novo plants.

Proposition 6 *For any type of firm, the distribution of productivity and plant sizes (by any measure) among newly grafted plants stochastically dominates the distributions of de novo plants.*

This result further implies that if high-q firms put more effort into searching for plants in the market than into de novo plant creation, they will end up buying more high-type plants in the market and therefore will own higher-type plants. The conditions under which they search more depend on the shape of the search and plant creation cost function, $c(\cdot)$, as well as the arrival rates of de novo plants and grafting possibilities, among other parameters. We do not provide sufficient conditions. Instead, we simply assume that the combination of fundamentals is such that higher-q firms invest more in searching than plant creation.

Assumption 5 *Parameters are such that higher q firms have higher ratios of search effort to de novo effort. Namely, $s(q)/n(q)$, is increasing in q.*

Under this assumption, we can show that the higher-q firms operate relatively more higher-a plants. Namely, the mass of plants of a given type is log-supermodular in the firm's type. The next proposition presents this result.

Proposition 7 *Let $\psi(q,a)$ be the measure of plants of type a operated by firms of type q. The measure $\psi(\cdot)$ is log-supermodular.*

Note that this result implies that, when a firm buys plants from another firm, it is more likely to find high-type plants in high-type firms. The best firms will buy plants from other good firms, since they are more likely to increase their effective productivity. Positive sorting between firms is the implication of the type of selection we demonstrated above. This is why the market for plants is, essentially, a market in which high-type firms buy the selection already made by previous firms: The market of selection.

3.6 Power-Law Tail Behavior

This section characterizes the behavior of the right tail of the size distribution of plants under the assumption that the distributions of firm and plant types follow power laws, too. Specifically, we make the following functional form assumptions.

Assumption 6 *Let $1-F_q(q) \sim q^{-\chi_q}$, $1-F_{a,0}(a) \sim a^{-\chi_a}$, and $\Phi_a \sim a^{\chi_\Phi}$. Let $h(a,q)$ be homogeneous of degree $\delta > 0$ with $h(1,\infty) < \infty$ and $h(\infty,1) < \infty$. Furthermore, let the shape parameters of the plant and firm distribution satisfy,*

$$\begin{aligned} \chi_a &> (1-\zeta)\,\chi_\Phi\ , \\ \chi_q &> \gamma\left[\zeta\delta - \chi_a + (1-\zeta)\,\chi_\Phi\right]\ . \end{aligned}$$

We define two constants that will govern the tail of the plant distribution:

$$\rho_1 \equiv \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{g_L}{\sigma^2/2}} \text{ , and } \rho_2 \equiv \zeta + \gamma\,(\zeta - 1) + \frac{\chi_q + (1+\gamma)\,(\chi_a - \zeta\delta)}{\chi_\Phi} \ .$$

Define also $G(x) \equiv \int G_a(x)da$ to be the measure of plants with productivity weakly less than x and $G^{new}(x)$ to be the measure of newly created plants with productivity weakly less than x. Then, we obtain the following result.

Proposition 8 *The cross-section of plants has the following properties:*

1. *Among type a plants, the measure with effective productivity larger than x has a power-law tail given by*
$$\lim_{x\to\infty} \frac{\ln\left[G_a(\infty) - G_a(x)\right]}{\ln x} = -\min\{\zeta, \rho_1\} \ ;$$
2. *Under Assumption 6, among all plants, the measure with effective productivity larger than x has a power-law tail given by*
$$\lim_{x\to\infty} \frac{\ln\left[G(\infty) - G(x)\right]}{\ln x} = -\min\{\zeta, \rho_1, \rho_2\} \ ;$$
3. *Under Assumption 6, among newly created plants, the measure with effective productivity larger than x has a power-law tail given by*
$$\lim_{x\to\infty} \frac{\ln\left[G^{new}(\infty) - G^{new}(x)\right]}{\ln x} = -\min\left\{\zeta, \rho_2\right\} \ .$$

We underscore three implications of Proposition 8. First, within a given plant type a, the right tail is governed either by the thickness of the tail of initial match productivity draws, ζ, or by the fluctuations in match productivity summarized by ρ_1. The latter coincides with the upper-tail behavior observed in models such as Luttmer (2007), where productivity follows a geometric Brownian motion. Intuitively, conditional on reaching a very high productivity x, the arrival rate of grafting events becomes negligible, and the Brownian component alone governs the tail. The observed tail exponent is therefore the smaller of ζ and ρ_1 (i.e., the thicker of the two implied tails).

Second, after aggregating across plant types, a third force can determine the upper tail, namely, the distribution of the assortative component, $h(q,a)$. Consider the set of plants with productivity larger than some productivity x. As x rises, the set of plants is more selected, consisting of plants and firms with better types. Under Assumption 6, endogenous selection across types generates an additional candidate tail exponent, ρ_2, so that the aggregate tail is governed by $\min\{\zeta, \rho_1, \rho_2\}$.

Finally, among newly created plants, there are two candidate determinants of the right tail. The tail can be driven either by unusually high initial match-productivity draws (captured by ζ) or by selection into extreme plant types and firm types (captured by ρ_2). Accordingly, the upper tail among entrants is governed by $\min\{\zeta, \rho_2\}$.

Note that, if ζ and ρ_1 are sufficiently large, then both the cross-sectional distribution of all plants and the distribution of entrants will share the same tail exponent, ρ_2. We show below that this is, in fact, the relevant case in the data.

3.7 Efficiency

In this section, we analyze the efficiency properties of the equilibrium allocation. The model we have proposed implies that in equilibrium, the value of plants, the value of entry, and the firm's search decisions satisfy the following three conditions:

$$rv_a(x) = w^{-\frac{1}{\alpha}}x - \Phi_a + \mu x v_a'(x) + \frac{\sigma^2}{2}x^2 v_a''(x) + (1-\beta)b_a \underline{x}_a^{\zeta} \int_x^{\infty} [v_a(u) - v_a(x)]\,\zeta u^{-\zeta-1}du \tag{25}$$

$$r\Phi^E = \int \left\{ \begin{array}{c} -c\left(n\left(q\right)\right) + \Upsilon n\left(q\right) \int_{\underline{x}_a}^{\infty} \int_{\underline{x}_a}^{x} v_a\left(x\right)\zeta x^{-\zeta-1} h\left(q,a\right)^{\zeta} dx dF_a(a) \\ -c(s(q)) + \beta \int_{\underline{x}_a}^{\infty} \int_{\underline{x}_a}^{x} \left[v_a\left(x\right) - v_a\left(u\right)\right]\zeta x^{-\zeta-1} h\left(q,a\right)^{\zeta} dG_a(x) da \end{array} \right\} dF_q(q) \tag{26}$$

$$c'\left(s(q)\right) = \frac{M(1,\theta)}{\theta} \int \beta h\left(q,a\right)^{\zeta} \int_{\underline{x}_a}^{\infty} \int_{\underline{x}_a}^{x} \left[v_a\left(x\right) - v_a\left(u\right)\right] dG_a(u)\zeta x^{-\zeta-1} dx da \tag{27}$$

In Appendix F we formulate the problem of a planner that maximizes the present value of the household's utility. In the steady state of the planner's allocation, all the equilibrium relationships hold except for those corresponding to the three conditions above. If we denote the efficient allocation with (*), we can show that in the planner's allocation, the corresponding relationships satisfy

$$rv_a^*(x) = \left(w^*\right)^{-\frac{1}{\alpha}} x - \Phi_a + \mu x v_a^{*\prime}(x) + \frac{\sigma^2}{2}x^2 v_a^{*\prime\prime}(x) + b_a^*(\underline{x}_a^*)^{\zeta} \int_x^{\infty} [v_a^*(u) - v_a^*(x)]\,\zeta u^{-\zeta-1}du \tag{28}$$

$$r\Phi^E = \int \left\{ -c\left(n^*(q)\right) - c\left(s^*(q)\right) + \Upsilon n^*(q) \int_{\underline{x}_a^*}^{\infty} \int_{\underline{x}_a^*}^{x} v_a^*(x)\zeta x^{-\zeta-1} h(q,a)^{\zeta} dx dF_a(a) \right\} dF_q(q) \tag{29}$$

$$c'(s^*(q)) = \frac{M(1,\theta^*)}{\theta^*} \int \left\{ h(q,a)^{\zeta} - \left(1 - \frac{\theta^* M_S(1,\theta^*)}{M(1,\theta^*)}\right) \overline{h(\cdot,a)^{\zeta}} \right\} \int_{\underline{x}_a^*}^{\infty} \int_{\underline{x}_a^*}^{x} \left[v_a^*(x) - v_a^*(u)\right] dG_a^*(u)\zeta x^{-\zeta-1} dx da \tag{30}$$

where $\theta^* \equiv \frac{S^*}{J^*} = \int s^*(q) dF_q(q)$ and $\overline{h(\cdot,a)^{\zeta}} \equiv \frac{\int h(q,a) s^*(q) dF_q(q)}{\int s^*(q) dF_q(q)}$ denotes the average suitability of searchers for type-a plants.

We can then assess the sources of inefficiency in our economy by inspecting the differences in these three equations. We identify three sources of inefficiencies:

1. Incumbent owners shut down plants too early. (25) and (28) align only if buyers have no bargaining power, $\beta = 0$. If $\beta > 0$, owners only capture a share of the gains from possible future grafting events. In contrast, the planner values the full surplus of keeping the plant operating.

2. There is too much entry. Comparing (26) and (29) reveals that in equilibrium, the value of entry includes the value of searching and finding targets. In contrast, the planner does not value this, as an additional firm is both a searcher and a target, so there are no gains from having more firms searching. Of course, this is an implication of the assumption that the matching function has constant returns. Because of this assumption, both the rate at which plants are encountered, which is proportional to $\frac{M(J,S)}{J} = M(1,\theta)$ and the rate at which searchers find plants, which is proportional to $\frac{M(J,S)}{S} = \frac{M(1,\theta)}{\theta}$, are independent of J.

3. Relative to the equilibrium, the planner would tilt search toward the high-q firms. Compare (27) and (30). The planner values the additional matches that come from search effort, net of the congestion externality, $h(q,a)^\zeta - \left(1 - \frac{\theta M_S(1,\theta)}{M(1,\theta)}\right) \overline{h(\cdot,a)^\zeta}$. In contrast, in equilibrium, searchers value only their share of the gains from a match, $\beta h(q,a)^\zeta$. These align only if there is no congestion ($\frac{\theta M_S(1,\theta)}{M(1,\theta)} = 1$) and buyers have all of the bargaining power, $\beta = 1$. With congestion, the planner would prefer to tilt search effort toward the high-q firms. The overall level of search might be too high or too low, depending on bargaining power and congestion levels.

Efficiency can then be naturally restored by correcting these three margins. In particular, it can be corrected using (i) a plant-type-specific subsidy to the operating cost of a plant, (ii) a tax on firm entry, and (iii) a firm-type-specific tax/subsidy on search.

4 Counterfactuals

4.1 Calibration

We calibrate the model in Section 3 using some of the facts presented in Section 2, together with a few additional moments we compute from the same data.

We rely on Proposition 3 to characterize the tail of the match productivity draws, ζ. Table 10 shows that after being grafted, a plant grows by between 3 and x.x%. Since the model is cast in continuous time, we select the result that compares the size of a plant one year before being grafted to their size one year after being grafted, as this is most consistent with a jump process. Therefore, we set $1/\zeta = 0.0xx$, or $\zeta = 1/0.0xx = xx.xx$.

We directly calibrate the drift, μ, and volatility, σ^2, of the Brownian motion process using growth rates in the cross section of establishments. We focus on employment dynamics of plants with 100 or more employees to account for selection.[19] We find that the average employment growth of these plants is $-x.xx\%$, and the standard deviation of their employment growth is xx.xx%. We therefore set $\mu = -x.xxx$ and $\sigma = x.xxx$.

[19] As Rossi-Hansberg and Wright (2007) show, small plants have higher growth rates largely due to their high exit rates; in other words, the growth rates are capturing the selection of the surviving plants that is not present for large plants that exit less frequently.

Next, we specify functional forms for several key functions in our model. First, we specify $h(q,a) = h_0(q^\omega + (a/a_0)^\omega)^{1/\omega}$, where a_0 is a shifter that allows us to vary the importance of the plant type. In our calibration, we set $\omega = -0.85$, $h_0 = 0.1$, and $a_0 = 3$ for now. Second, we assume that fixed costs take the form

$$\Phi_a = \underline{\Phi} + (\overline{\Phi} - \underline{\Phi})\left(\frac{a - \underline{a}}{\bar{a} - \underline{a}}\right)^{\chi_a}.$$

This specification allows us to flexibly control the scale, shape, and slope of fixed costs across plant types. We set $\chi_\Phi = 0.8$, $\underline{\Phi} = 1$, and $\overline{\Phi} = 7$ for now.

We assume a constant returns to scale matching function, $M(S,J) = \mathcal{M}S^{1-\eta}J^\eta$. We set the congestion elasticity to $\eta = 0.5$ and the scale of the function to $\mathcal{M} = 700$. For de novo plant creation, we set an arrival rate scale of $\Upsilon = 2{,}000$. We assume that plant creation (and search) costs are iso-elastic with parameter $\gamma = 0.25$, namely, $c(x) = x^{1+1/\gamma}/(1+1/\gamma)$.

Plant types, a, and firm types, q, are distributed according to a truncated Pareto distributions, so

$$F_a(a) = \frac{1 - \underline{a}^{\chi_a} a^{-\chi_a}}{1 - \underline{a}^{\chi_a} \bar{a}^{-\chi_a}} \text{ , and } F_q(q) = \frac{1 - \underline{q}^{\chi_q} q^{-\chi_q}}{1 - \underline{q}^{\chi_q} \bar{q}^{-\chi_q}} .$$

We set $\underline{a} = 6$, $\bar{a} = 100$, and $\chi_a = 0.2$, and $\underline{q} = 1$, $\bar{q} = 100$, and $\chi_q = 0.5$. Several of these parameters, such as χ_q and χ_a, are set to help match the tail parameter of the plant size distribution, which we estimate to be around x.x. Finally, we set a discount rate of $r = 0.05$, a bargaining parameter $\beta = 0.9$, a labor share $\alpha = 0.65$, and a firm entry cost $\Phi^E = 50$. For now, these parameter values are chosen to help us illustrate the model's properties clearly.

4.2 The Simulated Market for Plants

We present the results for plant-level and firm-level outcomes of our calibration in panels (a) to (f) and (g) to (i) of Figure 4, respectively. In our calibration, exit thresholds increase with plant type (a), indicating that higher-a plants have effective productivity distributions shifted to the right. Ultimately, this implies that higher-a plants are also, on average, larger.

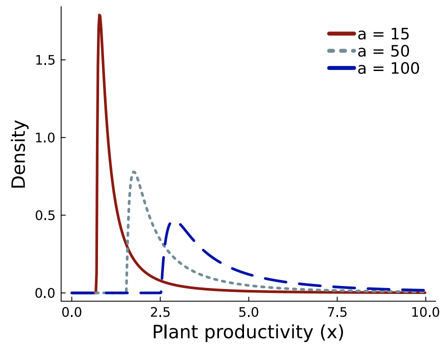
High-a plants also tend to be larger because they are more likely to be grafted. Aggregating across plant types, plants that have been grafted in the past are 15.6% larger than plants that have never been grafted. This is lower than our estimates of 28.5%-41% in Table 6, which is likely due to our high ζ estimate. In general, about 25% of plants in our economy are grafted at some point in time, which is also a bit higher than the same moment in the data. Grafting and assortative matching between plants and firms lead to a plant size tail parameter of about -1.86, a bit larger in absolute value than in the data.

Turning to firms (panels g-i), we find that high-q firms exert more effort in both search and de novo plant creation, with a relative bias toward search effort. Low-q firms, on the other hand, bias their effort toward de novo plant creation. Because most grafting happens among high-a plants, the relative plant creation intensities leads to sorting between plant types and firm types. In particular, higher-q firms' plant

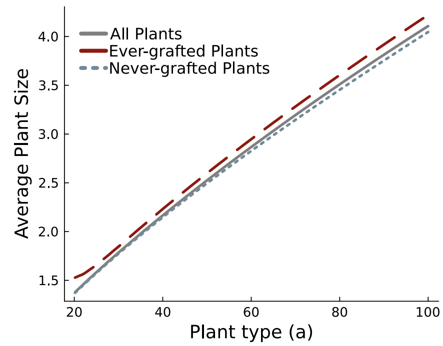
Figure 4: Calibration Results

A. Plant Statistics

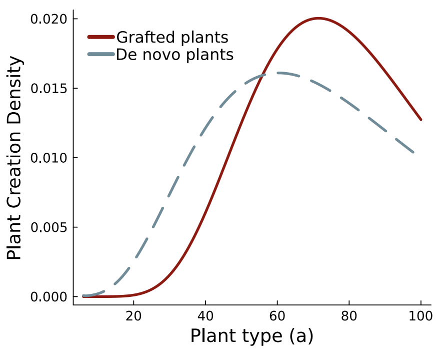
(a) Exit Thresholds



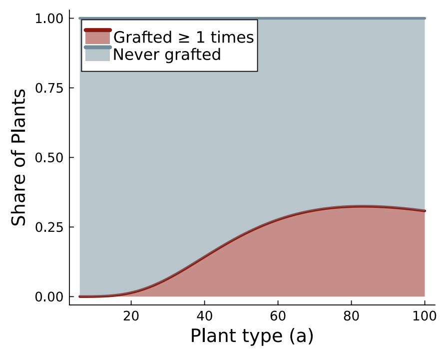
(b) Plant productivity distributions



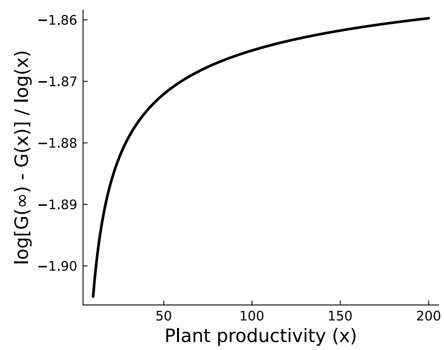
(c) Average plant size



(d) Plant creation density

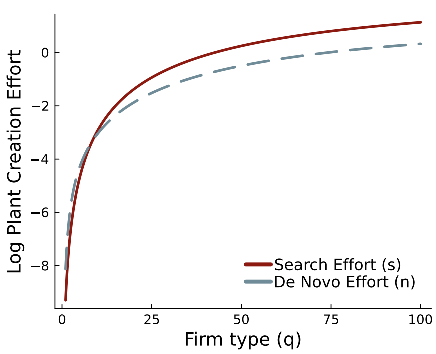


(e) Ever-grafted shares within a

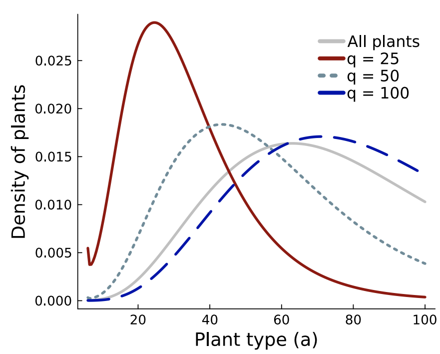


(f) Plant size distribution tail

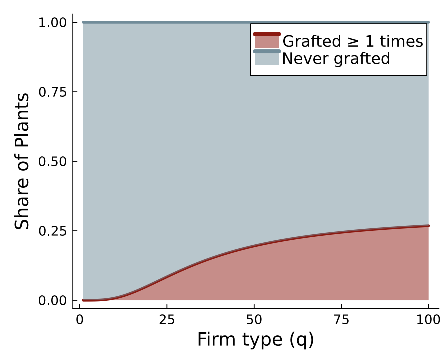
B. Firm Statistics



(g) Plant creation effort



(h) Sorting between plants & firms



(i) Ever-grafted shares within q

Notes. This figure displays various statistics from our calibrated model. The first panel conditions on plant types (a) while the second panel conditions on firm types (q).

type distributions are biased toward high-a plants, while lower-q firms' plant type distributions are biased toward low-a plants. Additionally, because higher-q firms are more likely to operate higher-a plants and exert more search effort, higher-q firms' plants are more likely to have been grafted in the past, consistent with our empirical findings.

Overall, the qualitative features of our modeled economy match the empirical description of the universe of plants and firms, and the market for plants, well. Quantitatively, the model, as specified and parametrized, generates too much grafting and not enough dispersion in plant productivities, given the high tail parameter that we estimate to match the relatively small observed increase in employment upon grafting.

4.3 Assessing the Role of Grafting

We first use the model to understand how grafting affects the allocation of plants across firms and what this implies for economic activity in equilibrium. To do so, we compare our simulated economy with a counterfactual economy in which we shut down grafting. We achieve this mechanically by setting $\mathcal{M} = 0$.

Figure 5 displays the results of this exercise. As the theory shows, grafting shifts the distribution of plants in equilibrium toward the more productive, high-a plants, which differentially impacts high-q firms. Removing grafting, therefore, shifts the distribution of plants toward middle- and low-a plants, as shown in panel (a). As a result, average plant sizes decline, but mostly for high-q firms (panel b). Since grafting favors the high-q firms, economic activity shifts from high-q firms toward middle-q firms (panel c).

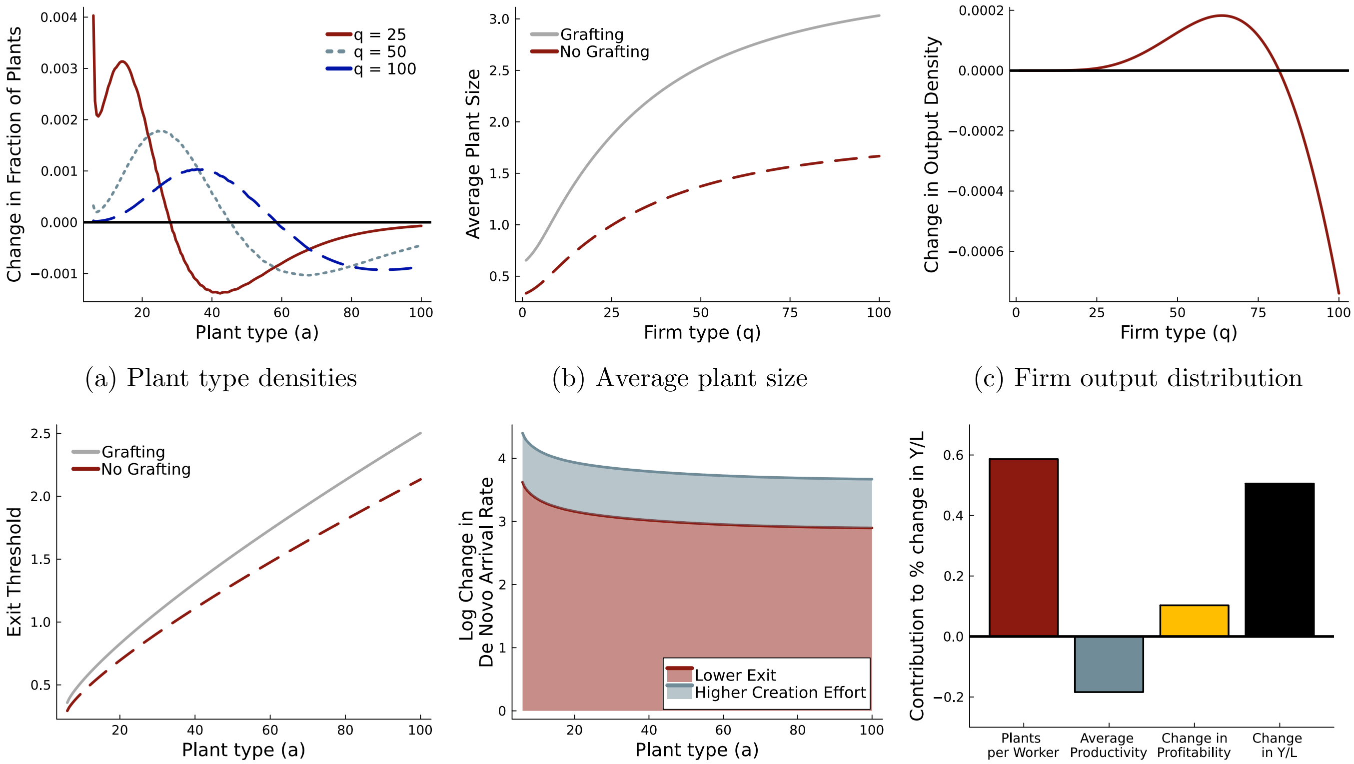
All else equal, one would expect that grafting reduces overall economic activity. This need not be the case in equilibrium due to the inefficiencies underscored in the previous section. When grafting is removed from the model, an important component of the firm's value, $u(q)$, declines relative to the grafting equilibrium. Wages, therefore, need to decline to encourage entry. When the elasticity of equilibrium wages to grafting is high enough, this can induce a large improvement in profitability for active plants. As such, exit thresholds may decline, which happens in our calibration (panel d).

The decline in exit thresholds has important implications for plant creation. The arrival rate of de novo plants is proportional to $\underline{x}_a^{-\zeta}$, so when the distribution of productivity draws is sufficiently thin-tailed, even small declines in exit thresholds can induce large inflows of plants (panel e). These inflows lead to a large increase in the equilibrium number of active plants in the economy, which, because the equilibrium is inefficient, can actually increase output per worker if the effect is strong enough to counteract the decline in average plant productivity. This, combined with improvements in output due to the lower wage rate, results in increases in output per worker, even when average productivity is declining due to the lack of grafting (panel f).

4.4 Assessing the Role of Assortative Matching

In our next counterfactual, we assess the effect of assortative matching in order to understand its role in determining aggregate productivity and plant creation through the implied sorting.

Figure 5: The Effects of Shutting Down Grafting



(a) Plant type densities

(b) Average plant size

(c) Firm output distribution

(d) Exit Thresholds

(e) De Novo Arrival Rate

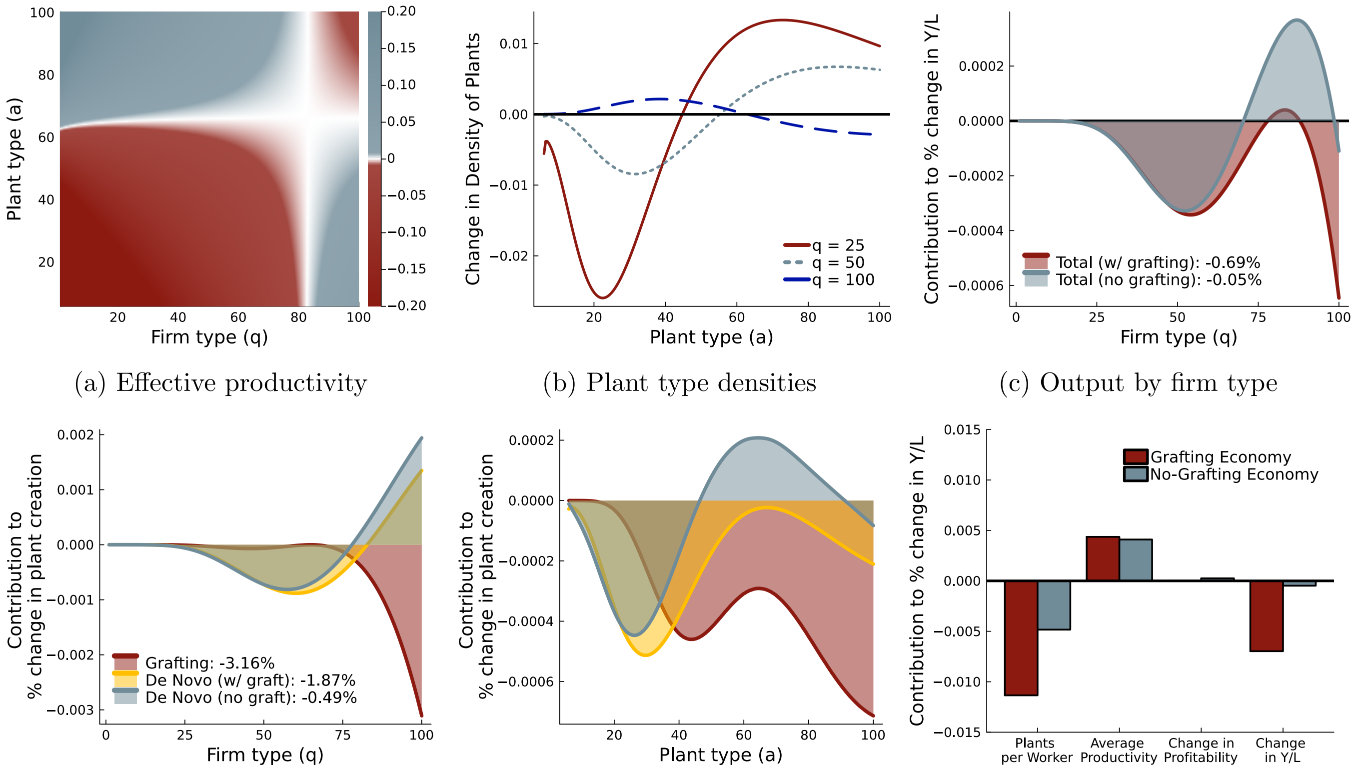
(f) Output by component

Notes. This figure shows the effects of removing grafting from the model. All figures display the the no-grafting economy relative to the grafting economy. Panel (a) reports the effects on the allocation of plant types across firm types, panel (b) reports the effects on average plant size across firm types, panel (c) reports how the distribution of output across firm types changes, panel (d) reports how exit thresholds change, panel (e) reports how de novo arrival rates change, broken down into an exit threshold component and a creation effort component, and panel (f) reports how different components contribute to changes in output.

Assumption 1 — i.e., that plants produce using a technology that is log-supermodular between the plant's type and its operating firm's type — drives assortative matching between plants and firms. In order to assess its impact, without changing the effective technology possibilities of the economy, we construct a set of firm-plant-specific transfers that incentivize firms to behave as if their productivity is log-modular. The chosen tax scheme is revenue-neutral—or output-neutral given that firms are price-takers in the goods market—for both plant and firm types. The actual production function retains the assumed log-supermodularity. For details, see Appendix G.

We report the results of this counterfactual in Figure 6. Panel (a) shows that when firms behave as if they have a log-modular technology, the effective productivity of firm-plant matches in our calibrated economy changes significantly. The lack of log-supermodularity reallocates plants in a way that reverses firm-plant sorting. Hence, high-q plants now operate more low-a plants, and low-q firms now operate more high-a plants (panel b).

Figure 6: The Effects of Removing Assortative Matching Between Firms and Plants



(a) Effective productivity (b) Plant type densities (c) Output by firm type

(d) Plant creation by firm type (e) Plant creation by plant type (f) Output by component

Notes. This figure shows how the grafting and no-grafting economies change when assortative matching is removed using log modular distortions. Panel (a) reports the effects on plant-firm productivity, panel (b) reports the effects on the distribution of plant types across firm types, panel (c) reports how output responds across firm types, panel (d) reports how plant creation effort is affected by firm type, panel (e) repeats panel (d) but across plant types, and panel (f) displays how different components affect output.

Output declines for virtually all firm types. High-q firms contribute the most to the decline, although the contributions are non-monotonic in q. In particular, because moderately high q firms (e.g., $q \approx 80$) are only marginally distorted by the change in incentives, they are less affected by the change. High-q firms operate worse plants on average, and since high-q firms have out-sized contributions to aggregate output per worker, plant productivity in the aggregate declines, driving down aggregate output per worker. However, low-q firms — of which there are many — now operate higher-a plants, which pushes aggregate plant productivity in the opposite direction (panel f). In the current calibration, the second effect dominates.

To build intuition for this result, we conduct the same experiment for an economy in which there is no grafting, which we achieve by setting the scale of the matching function, $\mathcal{M}$, to 0. We find only a negligible change in output per worker in the no-grafting economy, with moderately high-q firms actually experiencing a sharp increase in output per worker. In the aggregate, plant productivity increases modestly as before, but the sharp decline in plants per firm is absent.

This result suggests that in our simulated economy, the effect of assortative matching on output is not driven by an improved aggregate allocation of plants to firms, but rather, is driven by the effect on plant creation incentives. We therefore show how plant creation — both de novo and through grafting — changes across firm types (panel d) and plant types (panel e). De novo plant creation declines in the aggregate. The decline is driven by middle-q firms and partially offset by high-q firms. The intuition is that middle-q firms are now penalized when operating low-a plants, which have the highest arrival rates. As a result, they are more likely to shut these plants down before they are officially born, so fewer plants are created. In contrast, high-q firms now have an incentive to create plants, since the distortions subsidize their operation of low-a plants. Overall, because there are relatively few high-a plants in the economy, the decline in de novo plant creation by the low- and middle-q firms outweighs the increase in de novo creation by high-q firms.

The decline in de novo plant creation has ramifications for grafting. Since fewer plants are being created, firms operate fewer plants on average. But the decision to search for plants to graft depends on the likelihood of finding a good *plant-level* match conditional on matching with a firm; if firms have fewer plants, then the likelihood of finding such a match is lower. Therefore, search effort decreases. This effect is exacerbated for high-q firms because, as we showed in panel (d) of Figure 4, the distribution of grafted plants in the super-modular economy is concentrated among high-a plants, which high-q firms disproportionately operate. This effect is highlighted in panel (e) of Figure 6, which shows that, across plant types, high-a plants drive the majority of the aggregate decline in grafting even though low-a plants drive the majority of the decline in de novo plant creation.

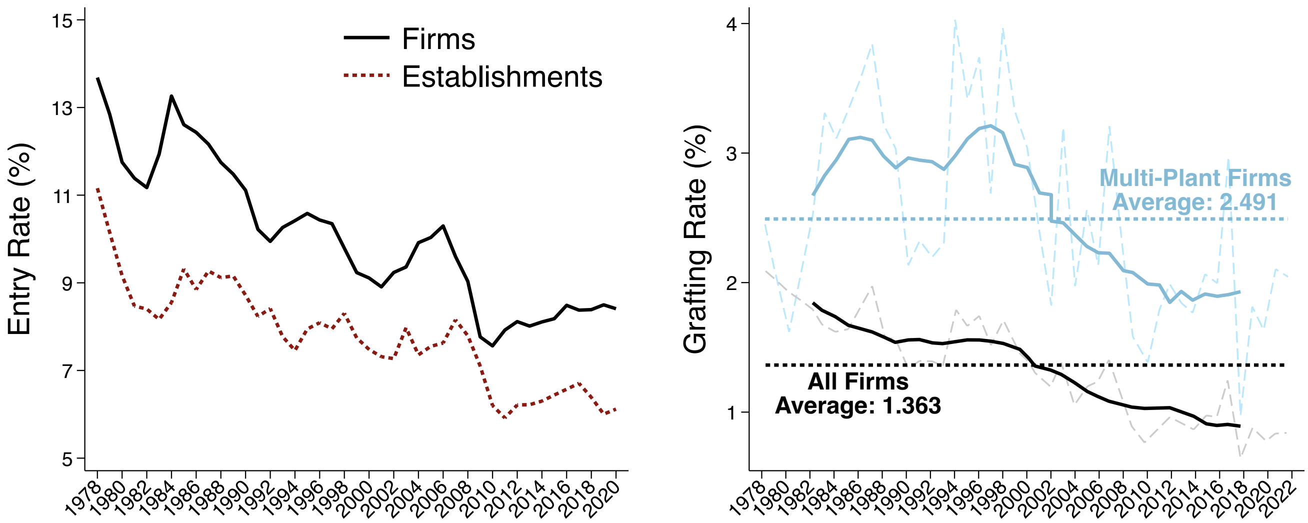
The reduction in grafting creates a unique feedback loop in our setting. Part of the value of de novo plant creation stems from the possibility of the plant being grafted in the future. Since search effort declines in the aggregate, plant creation becomes even less valuable relative to the no-grafting economy, which further contracts de novo creation effort. This is why the aggregate reduction in de novo plant creation is so pronounced relative to the no-grafting economy: the existence of grafting — and specifically, its role in plant-to-firm sorting — amplifies the first-order effect of the log-modular distortions.

4.5 Grafting, Dynamism, and Declining Population Growth

It is well known that, as we show in Figure 7 panel (a), both firm and establishment entry rates have persistently declined since the late 1970s. Karahan et al. (2024)shows that the decline in entry can in part be explained by declining population rates. In turn, this decline has been accompanied by a decline in grafting rates: in 1978, the grafting rate was just above 2%, while in 2022, it was only 0.8% (Figure 7 panel (b)).

Our theory naturally links these secular trends as potential outcomes of the decline in population growth rates. As population growth declines and entry rates fall, the availability of new establishments to graft (and new firms to graft from) also declines. As such, grafting became less viable as a means of expansion. How did this decline affect the allocation of plants across firms? What did it imply for concentration? And what is the impact on output? To answer these questions, we simulate the model with a 1 percentage-point lower population growth rate, moving from 3% to 2%. We compare the simulation outcomes to the model

Figure 7: Entry Rates and Grafting Rates Over Time



(a) Entry Rates

(b) Grafting Rates

Notes. This figure shows how entry and grafting rates have changed over time. Panel (a) plots entry rates for firms (black) and establishments (red) over time. Panel (b) reports grafting rates across all firms (black) and multi-plant firms (blue) over time. Raw time series are plotted in light dashed lines, while moving averages with an eight year window are reported in dark solid lines. Dotted horizontal lines reflect averages across time.

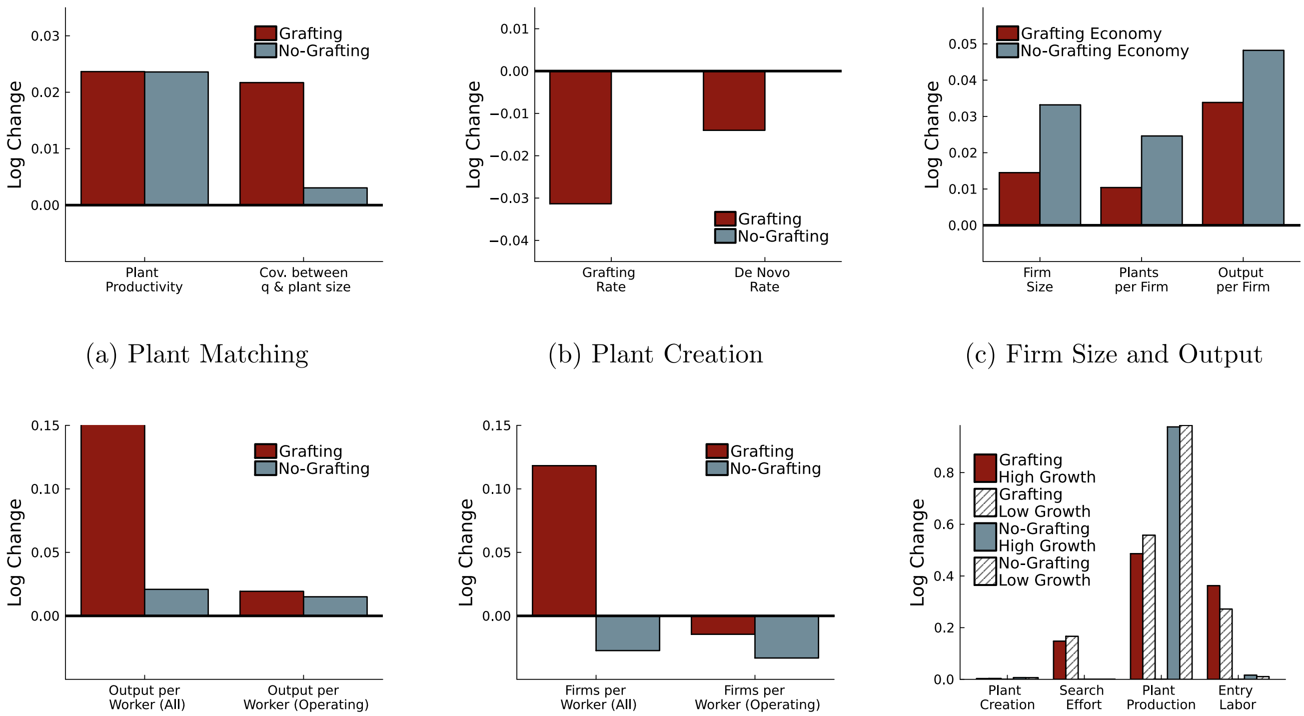
with no grafting, which serves as the benchmark.

Figure Figure 8 reports the results. In both the grafting and no-grafting economies, a decline in population growth increases the average age of firms. Therefore, in general, plants are better allocated across firms (panel a). The covariance between firm types and average plant size increases in both economies, but especially so in the grafting economy. However, the improved allocation of plants to firms reduces the incentive for firms to exert search effort, as they are less likely to find a viable plant to purchase. As such, grafting rates decline and, due to the feedback effect between grafting and de novo plant creation that we highlighted in Section Section 4.4, de novo plant creation declines as well (panel b).

The reduction in search and plant-creation efforts has ramifications for firm size. In both economies, due to the aging of firms, firm size (measured as non-entry labor divided by the number of firms) increases, firms have more plants, and each firm creates more output (panel c). However, due to the decline in plant creation in the grafting economy, the effects on firm size and plants per firm are attenuated substantially. As a result, output per firm responds less than in the no-grafting economy.

The implications for output *per worker* are more nuanced and ultimately depend on how we define the labor force (panel d). When we consider output relative to the entire labor force, output per worker increases substantially more than in the no-grafting economy. However, when we consider output relative to non-entry labor only — consisting of plant creation, search, and production labor — the effects on output

Figure 8: The Effects of a Decline in Population Growth



(a) Plant Matching (b) Plant Creation (c) Firm Size and Output

(d) Output per Worker (e) Firms per Worker (f) Labor Allocation

Notes. This figure shows how grafting and no-grafting economies change in response to a one percentage point decline in the population growth rate. Panel (a) reports the effects on plant-to-firm matches, panel (b) reports the effects on plant creation and search effort, panel (c) reports the effects on firm size and firm output, panel (d) reports the effects on output per worker using both the entire labor force and only non-entry labor, panel (e) repeats panel (d) but with firms per worker instead of output, and panel (f) reports the allocation of labor across plant creation effort, search effort, plant production, and entry costs.

per worker are similar across the two economies, with the grafting economy experiencing a 0.4 percentage point increase relative to the no-grafting economy.

When population growth declines, there are two key effects: there is less labor allocated to entry, and more labor allocated to production. The first effect increases the mass of firms in equilibrium, while the second effect decreases it. Thus, the effect on the total mass of firms depends on the relative importance of production and entry. In the grafting economy, entry labor is important enough to dominate the production labor component, whereas in the no-grafting economy, it is not. This explains why firms per worker move in opposite directions when we consider all labor, but not when we only consider non-entry labor.

5 Conclusion

The market for plants is large, active, and helps allocate the best plants to the best firms. Its existence and workings, in turn, affect firms' incentives to build new plants, since the value of these plants is not only determined by what the firm can produce with them, but also by their value if sold to grafting firms. Hence, the market for firms determines aggregate productivity, aggregate output, and the creation of firms and plants.

Despite its obvious importance, the market for plants has not been well studied in the economic literature. Most papers study firms, rather than plants, and mergers and acquisitions, rather than grafting. To bridge that gap, we empirically study the characteristics of this market and provide a parsimonious theory of heterogeneous firms that create heterogeneous plants and trade them in a market for plants. The model accounts well for the main empirical characteristics of the market we uncovered in the U.S. and lends itself to studying how different technological or demographic changes can affect the aggregate economy through the creation and allocation of plants across firms. We present some examples of these aggregate consequences, but many more could be analyzed. For example, in our theory, frictions in the market for plants have the potential to generate meaningful differences in income per worker. We hope that the empirically based theory of the market for plants we propose can prove useful for studying these and related issues.

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A Appendix

A.1 Data construction

In the following, we describe how the Census Bureau constructs its longitudinally-consistent establishment identifiers in the Longitudinal Business Database (LBD), i.e. `lbdnum`. The starting point for the LBD is the establishment-level master sampling frame of the Census Bureau, i.e., the Business Register (BR). When combined with information from their County Business Patterns (CBP) program, this becomes the County Business Patterns Business Register (CBPBR). This dataset serves as the core of the modern (or revised) LBD.

A.1.1 Construction of the CBPBR

This section describes how the basis files for the LBD, i.e. the BR's EINUNITS and EMPUNITS files, are constructed. When incorporating additional information from the CBP micro data files, the BR EMPUNIT files can be transformed into the CBPBR files. Our discussion is entirely based on DeSalvo et al. (2016), and Chow et al. (2021). The reader is referred to those working papers for more details on the BR and revised LBD. Information from the CBPBR comes from several sources, including IRS, SSA, BLS and Census Bureau. In the following, we will explain how each of these sources contributes to the Census Bureau's CBPBR files.

BR EINUNITS file. The starting point of the CBPBR is the Business Master File (BMF) from the IRS which essentially consists of the universe of Employer Identification Numbers (EINs). This data set is supplemented with information on employment, payroll, sales/receipts and detailed industry codes. Data on quarterly employment and payroll at the EIN level are sourced from IRS forms 941, 943 and 944 whereas annual data on sales/receipts are derived from IRS forms 1040 Schedule C (sole proprietors), 1120 (C corporations), 1120-S (S corporations), and 1065 (partnerships) and 990 (tax-exempt organizations, nonexempt charitable trusts, and section 527 political organizations). By construction, this will also indicate a tax unit's legal form of organization. Industry information (6-digit NAICS) is initially collected through these business income tax forms. Then, this skeleton structure of the BR (based on IRS' BMF) is updated with new EINs at a monthly frequency. Information on business births is supplied by the SSA through IRS form SS-4. This results in a data set containing the universe of businesses at the *tax* unit (EIN) level. This file is also referred to as the BR's EINUNITS file.

BR EMPUNITS file. However, the unit of interest is the level at which econonomic activity takes place at a physical location which will be referred to as an establishment (or plant). By construction, mapping tax units to establishments for single-establishment firms is simple. In fact, the firm, tax unit and establishment are identical in this case. The mapping from tax units to establishments is not straightforward for multi-establishment firms though. These firms are allowed to file employer and business income taxes for several establishments under one or multiple tax identifiers.

To split EINs into establishments, the Census Bureau uses several sources. Importantly, it receives establishment-level information from the BLS. The BLS compiles its own database of establishments., i.e. the Business Establishment List (BEL).[20]. In essence, the BLS is able to determine establishments from tax units through UI data. This is summarized by the following segments in Parker et al. (2001).

> "Its principal sources of information are the State Employment Security Agencies (SESA) of the fifty States and the District of Columbia (and Puerto Rico and the Virgin Islands). Employers report to the SESAs in compliance with State Unemployment Insurance (UI) laws, and for Federal civilian workers, in compliance with the Unemployment Compensation for Federal Employee (UCFE) program. Each quarter, business employers report monthly employment and quarterly wages."
>
> "Business enterprises with more than one establishment in a State are requested to file a Multiple Worksite Report (MWR) on which data for each of its establishments are reported separately. The EIN provides linkages among establishments of the same business enterprise across States. The EIN for establishments on the BEL is obtained from the initial Status Determination Form and updated, if necessary, based on the quarterly UI tax form."

Then, the BLS merges its BEL with a list of Census-supplied EINs and provides the Census Bureau with establishment-based information on location, industry classification codes, and ownership type. This is particularly helpful for new and small businesses (Fairman et al., 2008).

Finally, the Census Bureau also uses internal sources to disaggregate EINs into establishments. First, it relies on its results from the *Report of Organization* (ROO) survey which was previously known as the *Company Organization Survey* (COS). This is annual survey that the Census Bureau conducts to ask multi-unit companies about its establishment structure and operating information. Large firms are contained with certainty whereas smaller multi-unit firms are sampled.[21] According to DeSalvo, Klimek and Limehouse (2016), the ROO/COS surveys contain about 30 percent of multi-unit firms and a small amount of firms that have not been identified as a multi-unit entity but are likely to be so according to administrative records. As a result, the establishment structure of multi-unit firms is known with certainty for the largest businesses which is reassuring for our purposes of understanding the mechanics of grafting.

Second, the Economic Census, conducted every 5 years (ending in years with "2" and "7"), provides useful information. In addition to reporting sales/receipts at the establishment level, establishments also report under which EIN they reported business income taxes and provide information on operational status (e.g.,

[20] This data set serves as the BLS' main sampling frame for the Quarterly Census of Employment and Wages (QCEW)

[21] In the past, this certainty cutoff was 250 employees, which increased to 500 employees in 2013.

active/ceased and change in ownership). Third, the Census Bureau conducts the Annual Survey of Manufactures (ASM) which contains similar information on ownership and operational status but is conducted annually for manufacturing establishments. Given all these sources, the BR can be disaggregated to its primary unit, which is the establishment. Establishment identifiers in the BR EMPUNIT files were characterized by a Census File Number (`cfn`) before 2002. Since 2002, the BR uses the identifier `empunit_id`.[22]

In the above, we have described how to disaggregate tax units to the establishment level. However, we can also aggregate EINs up to the firm level. Once again, this is obvious for single-unit firms since tax units are equivalent to firms. For multi-unit enterprises, the Census Bureau assigns a 10-digit Enterprise ID which is a more general version of the historic 6-digit `alpha`. In the past, the Census Bureau would internally assign each multi-unit enterprise this ID and is still present/used in the revised LBD (see section 2). Information on firms is derived from IRS form 851 which is its Affiliations Schedule. In this form, corporations (i.e., those filing 1120 forms) report information on the common parent and its associated list of subsidiaries (which are each identified with an EIN). In addition, form 851 also indicates the parent's ownership share for each of its subsidiaries. For subsidiaries with multiple owners, the Census Bureau determines firm identifiers through majority ownership.

CBP microdata files. The CBP microdata files are bluntly speaking a cleaned-up version of the BR EMPUNITS files. In particular, establishment-level records with negative or zero payroll, and/or missing identifiers are dropped. As a result, it seems that these files would not contribute much (or anything for that matter) to the BR EMPUNITS files. However, the BR EMPUNITS (or SSEL SU and MU) files in the late 70s and early 80s were known to have some serious problems, including many missing establishments and missing values for crucial variables (e.g., employment). Fortunately, the Census Bureau was able to retrieve the CBP microdata files from 1976–1984 from an obsolete server.

This allowed the Census Bureau to retrieve some lost establishments in early years (i.e., those establishments that can only be found in either the CBP micro data or SSEL/BR files), and to assess the quality of employment and payroll variables for overlapping establishments. The integration of the CBP micro data files with the BR EMPUNITs files, and restricting this merged data set to employers (by dropping observations with missing identifiers and/or zero payroll figures) leads to the CBPBR files.

A.1.2 Longitudinal consistency

The establishment-level identifiers from the CBPBR files (`cfn` or `empunit_id_char`) are not necessarily longitudinally consistent. To deal with this, the Census Bureau carries out the following matching steps.

[22] Prior to 2002, the Census Bureau would refer to the BR's EINUNITS and EMPUNITS files as the SSEL SU and MU files, respectively.

1. Matching of existing establishment-level identifiers from year t to year $t+1$: cfn-to-cfn 1976–1977 to 2000–2001, cfn-to-empunit_id_char for 2001–2002, and empunit_id_char-to-empunit_id_char for 2002–2003 to the last available year. In the past (1982–2001), the Census Bureau created longitudinally-consistent links for some establishments through its permanent plant number (ppn). If it is available, information from ppn is exploited if no consecutive year matches were found through cfn or empunit_id_char.

2. By construction, establishment-level identifiers (such as cfn) would automatically change when a firm would change its multi-unit status. For example, cfn would be constructed through "EIN + 0" for an establishment belonging to a single-unit firm whereas it would equal "alpha + estabcounter" for establishments belonging to multi-unit firms.[23] To deal with this, the Census Bureau would longitudinally link EINs from year t to $t+1$. If some EIN in year $t+1$ would consist of one establishment, then the process would be finished. For multiple establishments belonging to some EIN in year $t+1$, the Census Bureau would consecutively look for a unique match on zip code, street address or name. If a unique match is still not found, then the Census Bureau employs "a string comparator function on name and street address to rank all potential matches and choose the highest ranked match" (Chow *et al.*, 2021).

3. Last, the Census Bureau looks for longitudinal links by comparing name and address fields across two consecutive years. This step is performed only for single-unit establishments though.

A.1.3 Longitudinally-consistent firm identifiers

By construction, firm-level identifiers (i.e., firmid and lbdfid) in Census data products are not longitudinally consistent over time. Thus, a change in lbdfid does not always reflect a change in ownership through, say, a merger/acquisition. The most problematic change is when an establishment transitions from single- to multi-unit status. Whenever an establishment is part of a single-unit firm, then firmid is constructed through the identity firmid $= 0 +$ EIN. However, this identify changes to firmid $= 0000 +$ alpha whenever it becomes a multi-unit organization. Note that alpha is an internal 6-digit identifier that the Census Bureau assigns to multi-unit organizations only. While lbdfid is not constructed in this way, it also suffers from mechanical firm ownership changes due to changes in multi-unit status.

Despite these issues, Dent et al. (2018) – henceforth, DPW – have identified some useful firm-level characteristics that allow us to distinguish between different types of ownership changes. To describe their method, let lbdfid1 and lbdfid2 loosely denote the lbdfid variable in years $t-1$ and t for some establishment (identified through lbdnum), respectively. Furthermore, the variables firm_firstyear_emp and firm_lastyear_emp denote the first and last year, respectively, when strictly positive employment was ob-

[23]The variable estabcounter consisted of 4 digits.

served at some `lbdfid`. The cases described below will guide us in our definition for grafting.

Case 1. SU-to-SU transition

An establishment changes its `lbdfid` from `lbdfid1` to `lbdfid2` but it keeps being a firm with a single establishment in periods $t-1$ and t.

1. **Reorganization (case *E* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`.

2. **Legal form reorganization (case *L* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2` with a change in the establishment's legal form of organization (`lfo`).

3. **Take-over by single-unit incumbent who lost previous establishment(s)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`.

4. **Simultaneous expansion and take-over by entrant**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`.

5. **Simultaneous expansion and take-over by incumbent who lost previous establishment(s)**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`.

Discussion. These SU-to-SU cases are hardest to understand since it is difficult to distinguish reorganizations from changes in ownership. Furthermore, some cases can be interpreted as relocations. For example, cases 1.3 and 1.5 can describe an incumbent business moving to a different location. In case 1.3, the target establishment closes whereas in case 1.5 it continues in a different location. If a relocation does not involve a name change, then we could explicitly verify with the Business Register whether cases 1.3 and 1.5 are, in fact, relocations. If that is the case, then these events would not be labeled as grafting. Lastly, it could be argued that case 1.4 is an unlikely grafting event. The entering firm takes over an existing establishment from a firm with only that specific establishment.

Case 2. SU-to-MU transition

An establishment changes its `lbdfid` but `lbdfid1` is a single-unit enterprise whereas `lbdfid2` is a multi-unit one. Then, there are four relevant cases:

1. **Establishment gets taken over by existing multi-unit enterprise (case *A2* in DPW).**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`. Some of these cases need to be corrected manually though. Establishments belonging to "small" multi-unit enterprises (i.e., employment less than 500) do not have their ownership status registered in Census' annual *Report of Organization* in non-Census years.[24] Instead, they are assigned a separate `lbdfid`. During Census years, this ownership information gets updated. As a result, there might be an artificial amount of "acquisitions" occurring during these years. We can identify some of these cases by exploiting name fields in the Business Register (BR).

2. **Organic growth (case *O* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. Furthermore, all *other* establishments in the multi-unit organization `lbdfid2` are new in t, i.e. `firstyear_emp` $= t$ for these establishments.

3. **Establishment becomes part of a new, merged enterprise (case *A0* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. Furthermore, all other establishments (created before t) that are currently part of the new enterprise `lbdfid2` were single-unit firms $t-1$. For these particular `lbdfid`, we must also have `firm_lastyear_emp` $= t-1$. By construction, there are at least two distinct `lbdfid1`.

4. **Establishment becomes part of a new enterprise that exists of multiple, existing enterprises (cases *A1/C1* in DPW)**.

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. The other establishments of `lbdfid2` in t used to belong to at least one multi-unit enterprise. `lbdfid` with the largest payroll in $t-1$ is labeled as the "acquiring" firm (C1) whereas the others get acquired.

5. **Pathological case (case *D* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. The multi-unit indicator of `lbdfid2` is unity, but this firm consists of only a single establishment. This case should not be happening but does occur (rarely).

6. **Simultaneous expansion and take-over by multi-unit entrant**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`.

7. **Simultaneous expansion and take-over by multi-unit incumbent**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`.

Discussion. Under case 1.3, a set of single-unit establishments come together to form a new organization in year t. It could be argued that this is a grafting event since these establishments are under new ownership. These cases cannot be included in certain regressions, however, since characteristics of the "buyer" in year $t-1$ are not properly defined. We have a similar issue in case 1.4 if we do not assign an "acquiring" firm

[24] This survey used to be known as the *Company Organization Survey* (COS).

(with DPW code C1). Even if we do, then the question remains whether these specific establishments are subject to a grafting event. Lastly, we will exclude events as grafting whenever it involves an expansion or takeover of an entrant (case 2.6 which is like case 1.4).

Case 3. MU-to-MU transition

`lbdfid1` is a multi-unit enterprise whereas `lbdfid2` is also a multi-unit one. Then, there are six cases:

1. **Establishment gets taken over by existing multi-unit enterprise (partial; case *A3* in DPW).**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`. This condition holds for a *subset* of establishments belonging to `lbdfid1` in $t-1$.

2. **Establishment gets taken over by existing multi-unit enterprise (full; case *A4* in DPW).**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`. This condition holds for *all* of establishments belonging to `lbdfid1` in $t-1$.

3. **Multi-unit spin-off (case *S1* in DPW)**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. A set of establishments in `lbdfid1` spins off into a new enterprise.

4. **Multi-unit reorganization (case *R1* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. If `lbdfid2` consists of only one multi-unit enterprise in $t-1$ (i.e., a single `lbdfid1`), then this is considered a reorganization.

5. **Multi-unit establishment becomes part of a new, merged multi-unit enterprise**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. If `lbdfid2` consists of at least two multi-unit enterprises in $t-1$ (i.e., multiple `lbdfid1` transition into `lbdfid2`), then a takeover is involved.

6. **Multi-unit reorganization (case *S2*/*C2* in DPW)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. If `lbdfid1` gets broken down into multiple multi-unit organizations, then the one with the largest payroll in $t-1$ is considered to take the organization further as a "continuer" (C2). The other multi-unit organizations are considered spin-offs.

Discussion. For case 3.4, if *all* establishments in $t-1$ that belong to `firmid1` fall under the ownership of `firmid2` in t, then this could reflect a reorganization or a name change. However, we cannot rule out with certainty the scenarios in which an entrant takes over an incumbent multi-unit firm or a merger.

Case 4. MU-to-SU transition

`lbdfid1` is a multi-unit enterprise whereas `lbdfid2` is a single-unit one. Then, there are three cases:

1. **Establishment spins off existing multi-unit enterprise (case *S3* in DPW).**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`.

2. **Establishment spins off multi-unit enterprise that gets reorganized (case *S4* in DPW).**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. A set of *other* establishments from `lbdfid1` continue under a new `lbdfid` in t.

3. **Exiting firm breaks into multiple single-unit enterprises (case *S5* in DPW).**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. This condition holds for every surviving establishment belonging to `lbdfid1` in $t-1$. Furthermore, all of these establishments are single-unit enterprises in t. The single-unit enterprise with the largest payroll in $t-1$ is assumed to continue (C3) the existing firm whereas the other single-unit enterprises are spin-offs.

4. **Mechanical firm ownership change/organic shrinking**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $= t$ for `lbdfid2`. All *other* establishments in period $t-1$ were part of the same, multi-unit organization `lbdfid1` and have `lastyear_emp` $= t-1$.

5. **Simultaneous multi-unit breakdown and take-over by single-unit incumbent who lost previous establishment(s)**

   `firm_lastyear_emp` $= t-1$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`.

6. **Spin-off taken over by single-unit incumbent**

   `firm_lastyear_emp` $\geq t$ for `lbdfid1` and `firm_firstyear_emp` $\leq t-1$ for `lbdfid2`.

We have ignored cases in which `firm_lastyear_emp` for `lbdfid1` in period $t-1$ is strictly less than $t-1$. Strictly speaking, this should not be happening, but it happens sporadically.

A.1.4 Grafting

Case	DPW code	Clean grafting	Comment
SU-to-SU transitions			
1.1	E	51	Reorganization/entrant included
1.2	L	51	Reorganization/entrant included
1.3			Relocation of incumbent included
1.4			Entrant included
1.5			Relocation of incumbent included
SU-to-MU transitions			
2.1	$A2$	51	
2.2	O		
2.3	$A0$	51	
2.4	$A1$	51	
	$C1$	51	Continuer included
2.5	D		
2.6			Expansion of entrant included
2.7		51	
MU-to-MU transitions			
3.1	$A3$	51	
3.2	$A4$	51	
3.3	$S1$	51	Spin-off included
3.4	$R1$	51	Reorganization included
3.5	$S2$	51	Spin-off included
	$C2$	51	Continuer included
MU-to-SU transitions			
4.1	$S3$	51	Spin-off included
4.2	$S4$	51	Spin-off included
4.3	$S5$	51	Spin-off included
	$C3$	51	Continuer included
4.4			
4.5			Expansion of SU incumbent included
4.6		51	Spin-off included

B Theoretical Results

B.1 Mathematical Preliminaries

In this section we will make heavy use of Modified Bessel functions.

Modified Bessel functions of the first and second kind, respectively $I_{\varpi}(\cdot)$ and $K_{\varpi}(\cdot)$, are the two solutions to the homogeneous modified Bessel differential equation,

$$0 = -\left(\varpi^2 + t^2\right) y(t) + t y'(t) + t^2 y''(t)$$

We will mostly be concerned with the first kind, which can be defined according to the expansion

$$I_{\varpi}(t) = \left(\frac{1}{2}t\right)^{\varpi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}t\right)^{2n}}{n!\Gamma(\varpi + n + 1)} \tag{31}$$

The two also satisfy the Wronskian Identity

$$-K'_{\varpi}(x) I_{\varpi}(x) + I'_{\varpi}(x) K_{\varpi}(x) = \frac{1}{x} \tag{32}$$

Lemma B.1

$$\lim_{u \to 0} \frac{I_{\varpi}(u)}{\frac{2^{-\varpi} u^{\varpi}}{\Gamma(\varpi+1)}} = 1 \tag{33}$$

$$\lim_{u \to 0} \frac{\frac{I_{\varpi}(u)}{\frac{2^{-\varpi} u^{\varpi}}{\Gamma(\varpi+1)}} - 1}{u^2} = \frac{1}{4(1+\varpi)} \tag{34}$$

Proof. From the expansion (31), we have

$$\lim_{u \to 0} \frac{I_{\varpi}(u)}{\frac{2^{-\varpi} u^{\varpi}}{\Gamma(\varpi+1)}} = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}z\right)^{2n}}{n!\Gamma(\varpi + n + 1)} = 1$$

and

$$\frac{I_{\varpi}(u)}{2^{-\varpi} u^{\varpi}} = \frac{1}{\Gamma(\varpi + 1)} + \frac{\left(\frac{1}{2}u\right)^2}{\Gamma(2 + \varpi)} + \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}u\right)^{2n}}{n!\Gamma(\varpi + n + 1)}$$

or, rearranging and taking the limit,

$$\lim_{u \to 0} \frac{\frac{I_{\varpi}(u)}{\frac{2^{-\varpi} u^{\varpi}}{\Gamma(\varpi+1)}} - 1}{u^2} = \frac{1}{4(1+\varpi)}$$

■

Lemma B.2 *(Gronwall 1932)* $\frac{tI'_\varpi(t)}{I_\varpi(t)}$ *is strictly increasing,* $\varpi < \frac{tI'_\varpi(t)}{I_\varpi(t)} \leq \sqrt{\varpi^2+t^2}$*, and* $\sqrt{\varpi^2+t^2} - \frac{tI'_\varpi(t)}{I_\varpi(t)}$ *is strictly increasing in* t*.*

Proof. [Incomplete] Define $p(x) \equiv \frac{xI'_\varpi(x)}{I_\varpi(x)}$ is strictly increasing. To see this, note that $\lim_{x\to 0} p(x) = \lim_{x\to 0} \frac{xI'_\varpi(x)}{I_\varpi(x)} = \varpi$. In addition,

$$\begin{aligned} p'(x) &= \frac{I'_\varpi(x) + xI''_\varpi(x)}{I_\varpi(x)} - \frac{xI'_\varpi(x)^2}{I_\varpi(x)^2} = \frac{1}{x}\left[\frac{xI'_\varpi(x) + x^2 I''_\varpi(x)}{I_\varpi(x)} - p(x)^2\right] = \frac{1}{x}\left[\frac{\left(\varpi^2+x^2\right) I_\varpi(x)}{I_\varpi(x)} - p(x)^2\right] \\ &= \frac{1}{x}\left[\left(\varpi^2+x^2\right) - p(x)^2\right] \end{aligned}$$

Note that $\lim_{x\to 0} p'(x) = \lim_{x\to 0} x \lim_{x\to 0} \frac{1}{x^2}\left[\left(\varpi^2+x^2\right) - p(x)^2\right] = 0$, and that $\lim_{x\to 0} p''(x) > 0$:

$$\begin{aligned} p'(x) + xp''(x) &= 2x - 2p(x)\,p'(x) = 0 \\ p''(x) &= 2 - [2p(x)+1]\frac{p'(x)}{x} \\ \lim_{x\to 0} p''(x) &= 2 - 3\lim_{x\to 0}\frac{p'(x)}{x} \\ \lim_{x\to 0} p''(x) &= \frac{1}{2} > 0 \end{aligned}$$

So $p(0) > 0$, and there is an $\varepsilon > 0$ so that $p'(x) > 0$ for $x \in (0,\varepsilon)$. Toward a contradiction, suppose there is an x_0 such that $p(x_0) \leq 0$. Let $x_0 = \inf_{x>0}\{p'(x) = 0\}$. Thus it must be that p is increasing on $[0, x_0]$. At such a point, $p'(x_0) = 0$ which also means the numerator is zero, i.e., $\left(\tilde{\varpi}^2 + x_0^2\right) - p(x_0)^2 = 0$. However, the derivative of the numerator is $2x_0 - 2p(x_0)\,p'(x_0) = 2x_0 > 0$, so just above x_0, the numerator of $p'(x)$ is positive, and the denominator of $p'(x)$ is positive, thus $p'(x)$ is positive, meaning that x_0 is not in fact the infimum. ■

Lemma B.3 *For a constant* $c \in (-\varpi, \varpi)$*, Define* $J_{\varpi,c}(\lambda) \equiv \frac{\int_0^\lambda I_\varpi(t)t^{c-1}dt}{\lambda^c I_\varpi(\lambda)}$*.* $J_{\varpi,c}(\lambda)$ *is strictly decreasing in* λ *and satisfies the following two limits*

$$\lim_{\lambda\to 0} J_{\varpi,c}(\lambda) = \frac{1}{c+\varpi} \tag{35}$$

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\left(J_{\varpi,c}(\lambda) - \frac{1}{\varpi+c}\right) = -\frac{1}{4(\varpi+1)}\frac{2}{\varpi+2+c} \tag{36}$$

Proof. We first show that $J_{\varpi,c}(\lambda)$ is strictly decreasing. For $\lambda > 0$, changing variables gives

$$J_{\varpi,c}(\lambda) = \frac{\int_0^\lambda I_\varpi(t)\,t^{c-1}dt}{\lambda^c I_\varpi(\lambda)} = \int_0^1 \frac{I_\varpi(\lambda u)}{I_\varpi(\lambda)} u^{c-1} du$$

Next, we show that each $\frac{I_\varpi(\lambda u)}{I_\varpi(\lambda)}$ is decreasing in λ. Since $u<1$ and $\frac{xI'_\varpi(x)}{I_\varpi(x)}$ is increasing (from lemma)

$$\frac{d\log\frac{I_\varpi(\lambda u)}{I_\varpi(\lambda)}}{d\log\lambda}=\frac{\lambda u I'_\varpi(\lambda u)}{I_\varpi(\lambda u)}-\frac{\lambda I'_\varpi(\lambda)}{I_\varpi(\lambda)}<0$$

Together, these imply that $J_{\varpi,c}(\lambda)$ is strictly decreasing.

Next, we take the limit and use L'Hopital's rule to get

$$\begin{aligned}\lim_{\lambda\to 0}J_{\varpi,c}(\lambda) &= \lim_{\lambda\to 0}\frac{\int_0^\lambda I_\varpi(t)\,t^{c-1}dt}{\lambda^c I_\varpi(\lambda)}=\lim_{\lambda\to 0}\frac{I_\varpi(\lambda)\,\lambda^{c-1}}{c\lambda^{c-1}I_\varpi(\lambda)+\lambda^c I'_\varpi(\lambda)}=\lim_{\lambda\to 0}\frac{1}{c+\frac{\lambda I'_\varpi(\lambda)}{I_\varpi(\lambda)}}\\ &= \frac{1}{c+\varpi}\end{aligned}$$

To get at the higher order term, we use the expansion

$$I_\varpi(z)=\left(\frac{1}{2}z\right)^\varpi\sum_{n=0}^\infty\frac{\left(\frac{1}{2}z\right)^{2n}}{n!\Gamma(\varpi+n+1)}$$

The integral is thus

$$\begin{aligned}\int_0^\lambda I_\varpi(t)\,t^{c-1}dt &= \int_0^\lambda\left(\frac{1}{2}t\right)^\varpi\sum_{n=0}^\infty\frac{\left(\frac{1}{2}t\right)^{2n}}{n!\Gamma(\varpi+n+1)}t^{c-1}dt\\ &= \int_0^\lambda\sum_{n=0}^\infty\frac{2^{-\varpi-2n}t^{\varpi+2n+c-1}}{n!\Gamma(\varpi+n+1)}dt\\ &= \sum_{n=0}^\infty\frac{\frac{1}{\varpi+2n+c}2^{-\varpi-2n}\lambda^{\varpi+2n+c}}{n!\Gamma(\varpi+n+1)}\end{aligned}$$

We therefore can write

$$\begin{aligned}\frac{\int_0^\lambda I_\varpi(t)\,t^{c-1}dt}{\lambda^c I_\varpi(\lambda)}-\frac{1}{c+\varpi} &= \frac{\sum_{n=0}^\infty\frac{\frac{1}{\varpi+2n+c}2^{-\varpi-2n}\lambda^{\varpi+2n+c}}{n!\Gamma(\varpi+n+1)}}{\sum_{n=0}^\infty\frac{2^{-\varpi-2n}\lambda^{\varpi+2n+c}}{n!\Gamma(\varpi+n+1)}}-\frac{1}{c+\varpi}\\ &= \frac{\sum_{n=0}^\infty\frac{\frac{1}{\varpi+2n+c}2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}}{\sum_{n=0}^\infty\frac{2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}}-\frac{1}{c+\varpi}\\ &= \frac{\sum_{n=0}^\infty\frac{\frac{1}{\varpi+2n+c}2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}-\frac{1}{c+\varpi}\sum_{n=0}^\infty\frac{2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}}{\sum_{n=0}^\infty\frac{2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}}\\ &= \frac{\sum_{n=1}^\infty\frac{\frac{1}{\varpi+2n+c}2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}-\frac{1}{c+\varpi}\sum_{n=1}^\infty\frac{2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}}{\sum_{n=0}^\infty\frac{2^{-\varpi-2n}\lambda^{2n}}{n!\Gamma(\varpi+n+1)}}\end{aligned}$$

where the last line follows because the $n = 0$ terms cancel. Dividing by λ^2 and taking the limit, only the $n = 1$ terms in the numerator and $n = 0$ terms in the denominator remain:

$$\begin{aligned}
\lim_{\lambda \to 0} \frac{1}{\lambda^2} \left(\frac{\int_0^\lambda I_\varpi(t) t^{c-1} dt}{\lambda^c I_\varpi(\lambda)} - \frac{1}{c+\varpi} \right) &= \frac{\frac{\frac{1}{\varpi+2+c} 2^{-\varpi-2}}{\Gamma(\varpi+2)} - \frac{1}{c+\varpi} \frac{2^{-\varpi-2}}{\Gamma(\varpi+2)}}{\frac{2^{-\varpi}}{\Gamma(\varpi+1)}} \\
&= \frac{1}{4(\varpi+1)} \left(\frac{1}{\varpi+2+c} - \frac{1}{c+\varpi} \right) \\
&= -\frac{1}{4(\varpi+1)} \frac{2}{\varpi+2+c} \frac{1}{c+\varpi}
\end{aligned}$$

■

Another useful property will be the large t behavior of $I_\varpi(t)$,

$$\lim_{t \to \infty} \frac{I_\varpi(t)}{\frac{e^t}{\sqrt{2\pi t}}} = 1 \tag{37}$$

With this, one can derive the large t behvior of some other objects that will be useful:

Lemma B.4 *For any c, as t grows large*

$$\lim_{t \to \infty} \frac{\int_0^t u^c I_\varpi(u) du}{t^c \frac{e^t}{\sqrt{2\pi t}}} = 1$$

and

$$\lim_{t \to \infty} t J_{\varpi,c}(t) = 1$$

Proof. Using L'Hopital's rule, ■

$$\lim_{t \to \infty} \frac{\int_0^t u^c I_\varpi(u) du}{t^c \frac{e^t}{\sqrt{2\pi t}}} = \lim_{t \to \infty} \frac{t^c I_\varpi(t)}{\left(c - \frac{1}{2}\right) t^{c-1} \frac{e^t}{\sqrt{2\pi t}} + t^c \frac{e^t}{\sqrt{2\pi t}}} = \lim_{t \to \infty} \frac{1}{\left(c - \frac{1}{2}\right) \frac{1}{t} + 1} \frac{I_\varpi(t)}{\frac{e^t}{\sqrt{2\pi t}}} = 1$$

$$\lim_{t \to \infty} t J_{\varpi,c}(t) = \lim_{t \to \infty} t \frac{\int_0^t u^{c-1} I_\varpi(u) du}{t^c I_\varpi(u)} = \lim_{t \to \infty} \frac{\frac{e^t}{\sqrt{2\pi t}}}{I_\varpi(u)} \frac{\int_0^t u^{c-1} I_\varpi(u) du}{t^{c-1} \frac{e^t}{\sqrt{2\pi t}}} = 1$$

C The Value Function

This section characterizes the value function and the exit threshold. Recall that the value function satisfies the HJB

$$rv_a(x) = w^{-\frac{1}{\alpha}}x - \Phi_a + (1-\beta)\underline{x}_a^{\zeta} b_a \int_0^{\infty} \max\{0, v_a(u) - v_a(x)\}\zeta u^{-\zeta-1} du + \mu x v_a'(x) + \frac{\sigma^2}{2}x^2 v_a''(x)$$

along with the value matching condition, $v_a(\underline{x}_a) = 0$, the smooth pasting condition, $v_a(\underline{x}_a) = 0$, and a no-bubble condition. To get at the no bubble condition, suppose the firm kept the plant forever. If the plant's productivity at t where x, the expectation of its productivity at $t+\Delta$ would be $xe^{\mu\Delta}$, so that expected profit is $\int_0^{\infty} e^{-r\Delta}\left[w^{-\frac{1}{\alpha}}xe^{\mu\Delta} - \Phi_a\right]d\Delta = \frac{\bar{\pi}}{r-\mu}x - \frac{\Phi_a}{r}$. The no bubble condition is

$$\lim_{x\to\infty} v_a(x) - \left(\frac{w^{-\frac{1}{\alpha}}}{r-\mu}x - \frac{\Phi_a}{r}\right) = 0$$

Define the following variables:

$$\lambda_a \equiv \left(\frac{(1-\beta)b_a}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}\right)^{1/2}$$

$$k \equiv \frac{1}{\zeta}\left(1 + \frac{\mu}{\sigma^2/2}\right)$$

and the function

$$y_a(t) \equiv \frac{1}{w^{-\frac{1}{\alpha}}}\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2 t^{-k} v_a'\left(\underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}\right)$$

By differentiating the HJB with respect to x, one can verify that $y_a(t)$ satisfies the modified Bessel differential equation

$$-t^{-k} = -\left[\varpi^2 + t^2\right]y_a(t) + ty_a'(t) + t^2 y_a''(t) \tag{38}$$

where ϖ is the positive root of

$$\varpi^2 = k^2 + \frac{r-\mu}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}$$

with boundary conditions given by the smooth pasting condition and the derivative of the no bubble condition, $\lim_{x\to\infty} v_a'(x) = \frac{w^{-\frac{1}{\alpha}}}{r-\mu}$:

$$y_a(\lambda_a) = 0 \tag{39}$$

$$\lim_{t\to 0} t^k y_a(t) = \frac{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}{r-\mu} \tag{40}$$

Let $I_{\varpi}(\cdot)$ be the modified Bessel function of the first kind. Define the functions

$$\theta(\lambda) \equiv \frac{1}{\lambda I_{\varpi}(\lambda)^2} \int_0^{\lambda} u^{-k-1} I_{\varpi}(u)\, du$$
$$\Theta(x_1, x_2) \equiv \int_{x_2}^{x_1} \theta(u)\, du$$

Proposition C.1 *The solution to each ODE (38) subject to boundary conditions (39) and (40) is $y_a(t) = y(t, \lambda_a)$, where*

$$y(t, \lambda) \equiv I_{\varpi}(t)\, \Theta(\lambda, t)$$

Proof. The two linearly independent solutions are the modified Bessel functions of the first and second kind, $I_{\varpi}(t)$ and $K_{\varpi}(t)$. Variation of parameters yields the particular solution

$$y_{ap}(t) = -I_{\varpi}(t) \int_{\lambda_a}^{t} \tilde{t}^{-k-1} K_{\varpi}\left(\tilde{t}\right) d\tilde{t} + K_{\varpi}(t) \int_{\lambda_a}^{t} \tilde{t}^{-k-1} I_{\varpi}\left(\tilde{t}\right) d\tilde{t}$$

Putting the particular and homogeneous solutions together, we have that the solution is

$$y_a(t) = -I_{\varpi}(t) \left\{ C_a + \int_{\lambda_a}^{t} \tilde{t}^{-k-1} K_{\varpi}\left(\tilde{t}\right) d\tilde{t} \right\} + K_{\varpi}(t) \left\{ \tilde{C}_a + \int_{\lambda_a}^{t} \tilde{t}^{-k-1} I_{\varpi}\left(\tilde{t}\right) d\tilde{t} \right\}$$

for constants C_a and $\tilde{C}_a$. To find the two constants, we impose the boundary conditions. We first consider the no bubble condition. We are interested in $\lim_{t\to 0} t^k y(t)$. Note that as $t \to 0$, $I_{\varpi}(t) \sim \frac{1}{\Gamma(1+\varpi)} \left(\frac{1}{2}t\right)^{\varpi}$ and $K_{\varpi}(t) \sim \frac{1}{2}\Gamma(\varpi) \left(\frac{t}{2}\right)^{-\varpi}$. Consider first

$$\lim_{t\to 0} t^k K_{\varpi}(t) \left\{ \tilde{C}_a + \int_{\lambda_a}^{t} \tilde{t}^{-k-1} I_{\varpi}\left(\tilde{t}\right) d\tilde{t} \right\}$$

Since $t^k K_{\varpi}(t) \sim \Gamma(\varpi)\, 2^{\varpi-1} t^{k-\varpi}$ and $k - \varpi < 0$, $t^k K_{\varpi}(t)$ is explosive as $t \to 0$. Thus it must be that $\tilde{C}_a = \int_0^{\lambda_a} \tilde{t}^{-k-1} I_{\varpi}\left(\tilde{t}\right) d\tilde{t}$. Plugging this in gives

$$y_a(t) = -I_{\varpi}(t) \left\{ C_a + \int_{\lambda_a}^{t} \tilde{t}^{-k-1} K_{\varpi}\left(\tilde{t}\right) d\tilde{t} \right\} + K_{\varpi}(t) \int_0^{t} \tilde{t}^{-k-1} I_{\varpi}\left(\tilde{t}\right) d\tilde{t}$$

With this, one can verify the no bubble condition

$$\lim_{t\to 0} t^k y_a(t) = \frac{1}{\varpi^2 - k^2} = \frac{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}{r - \mu}$$

With this in hand, we next impose the smooth pasting condition, $y_a(\lambda_a) = 0$. This gives

$$0 = y_a(\lambda_a) = -I_\varpi(\lambda_a) C_a + K_\varpi(\lambda_a) \int_0^{\lambda_a} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t}$$

or

$$C_a = \frac{K_\varpi(\lambda_a)}{I_\varpi(\lambda_a)} \int_0^{\lambda_a} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t}$$

We thus have

$$\frac{y_a(t)}{I_\varpi(t)} = -\frac{K_\varpi(\lambda_a)}{I_\varpi(\lambda_a)} \int_0^{\lambda_a} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} - \int_{\lambda_a}^{t} \tilde{t}^{-k-1} K_\varpi(\tilde{t})\, d\tilde{t} + K_\varpi(t) \int_0^{t} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t}$$

Using $y_a(t) = y(t, \lambda_a)$, we have

$$\begin{aligned}
\frac{y(t,\lambda)}{I_\varpi(t)} &= -\frac{K_\varpi(\lambda)}{I_\varpi(\lambda)} \int_0^{\lambda} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} - \int_{\lambda}^{t} \tilde{t}^{-k-1} K_\varpi(\tilde{t})\, d\tilde{t} + \frac{K_\varpi(t)}{I_\varpi(t)} \int_0^{t} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} \\
&= -\frac{K_\varpi(\lambda)}{I_\varpi(\lambda)} \int_0^{\lambda} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} - \int_{\lambda}^{\infty} \tilde{t}^{-k-1} K_\varpi(\tilde{t})\, d\tilde{t} + \int_{t}^{\infty} \tilde{t}^{-k-1} K_\varpi(\tilde{t})\, d\tilde{t} + \frac{K_\varpi(t)}{I_\varpi(t)} \int_0^{t} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} \\
&= \int_t^{\lambda} \theta(u)\, du
\end{aligned}$$

where the last line uses the fundamental theorem of calculus and defining

$$\theta(u) \equiv \frac{d}{du}\left\{-\frac{K_\varpi(u)}{I_\varpi(u)} \int_0^{u} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} - \int_{u}^{\infty} \tilde{t}^{-k-1} K_\varpi(\tilde{t})\, d\tilde{t}\right\}$$

differentiating and using the Wronskian identity (32) gives

$$\begin{aligned}
\theta(u) &= -\frac{d}{du}\left\{\frac{K_\varpi(u)}{I_\varpi(u)}\right\} \int_0^{u} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} \\
&= \frac{-I_\varpi(u) K'_\varpi(u) + I'_\varpi(u) K_\varpi(u)}{I_\varpi(u)^2} \int_0^{u} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t} \\
&= \frac{1}{u I_\varpi(u)^2} \int_0^{u} \tilde{t}^{-k-1} I_\varpi(\tilde{t})\, d\tilde{t}
\end{aligned}$$

■

Claim C.2 *Define the function* $H(\cdot)$ *as*

$$H(\lambda) \equiv \lambda^{2/\zeta} \int_0^{\lambda} \left[1 - \left(\varpi^2 - k^2\right) t^k y(t,\lambda)\right] t^{-\frac{2}{\zeta}-1} dt$$

The exit threshold satisfies

$$\underline{x}_a = \frac{\frac{r-\mu}{r}\Phi_a w^{\frac{1}{\alpha}}}{1+\frac{2}{\zeta}H(\lambda_a)}$$

Proof. Using the fundamental theorem of calculus, taking the limit, and imposing the no-bubble condition give

$$\begin{aligned}
v_a(\underline{x}_a) - \left(\frac{w^{-\frac{1}{\alpha}}}{r-\mu}\underline{x}_a - \frac{\Phi_a}{r}\right) &= \lim_{\tilde{u}\to\infty} v_a(\tilde{u}) - \left(\frac{w^{-\frac{1}{\alpha}}}{r-\mu}\tilde{u} - \frac{\Phi_a}{r}\right) - \int_{\underline{x}_a}^{\tilde{u}} \left[v'(u) - \frac{w^{-\frac{1}{\alpha}}}{r-\mu}\right] du \\
&= \int_{\underline{x}_a}^{\infty} \left[\frac{w^{-\frac{1}{\alpha}}}{r-\mu} - v'(u)\right] du \\
&= \int_0^{\lambda_a} \left[\frac{w^{-\frac{1}{\alpha}}}{r-\mu} - v'\left(\underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}\right)\right] \frac{2}{\zeta}\underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}-1} dt \\
&= \int_0^{\lambda_a} \left[\frac{w^{-\frac{1}{\alpha}}}{r-\mu} - \frac{1}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2} w^{-\frac{1}{\alpha}} t^k y(t,\lambda_a)\right] \frac{2}{\zeta}\underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}-1} dt
\end{aligned}$$

Using the value matching condition $v_a(\underline{x}_a) = 0$ and $\varpi^2 - k^2 = \frac{r-\mu}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}$ and rearranging gives

$$\begin{aligned}
\frac{r-\mu}{r}\Phi_a w^{\frac{1}{\alpha}} &= \underline{x}_a \left(1 + \frac{2}{\zeta}\lambda_a^{\frac{2}{\zeta}} \int_0^{\lambda_a} \left[1 - \frac{r-\mu}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2} t^k y(t,\lambda_a)\right] t^{-\frac{2}{\zeta}-1} dt\right) \\
&= \underline{x}_a \left(1 + \frac{2}{\zeta}\lambda_a^{\frac{2}{\zeta}} \int_0^{\lambda_a} \left[1 - \left(\varpi^2 - k^2\right) t^k y(t,\lambda_a)\right] t^{-\frac{2}{\zeta}-1} dt\right)
\end{aligned}$$

■

We next derive some properties of the function H. In particular, we use a sequence of lemmas to show that H is increasing and $1 + \frac{2}{\zeta}H(\lambda)$ is log convex.

Lemma C.3 $\varpi + k - \frac{2}{\zeta} > 0$

Proof. Starting with the definitions $k \equiv \frac{1}{\zeta}\left(1 + \frac{\mu}{\sigma^2/2}\right)$ and $\varpi \equiv \sqrt{k^2 + \frac{r-\mu}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}}$, we can express ϖ as

$$\begin{aligned}
\varpi &= \sqrt{\left(\frac{1}{\zeta}\left(1+\frac{\mu}{\sigma^2/2}\right)\right)^2 + \frac{r-\mu}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}} = \frac{1}{\zeta}\sqrt{\left(1+\frac{\mu}{\sigma^2/2}\right)^2 + 4\frac{r-\mu}{\sigma^2/2}} \\
&= \frac{1}{\zeta}\sqrt{\left(\frac{\mu}{\sigma^2/2} - 1\right)^2 + 4\frac{r}{\sigma^2/2}}
\end{aligned}$$

Using $k - \frac{2}{\zeta} = \frac{1}{\zeta}\left(1 + \frac{\mu}{\sigma^2/2}\right) - \frac{2}{\zeta} = \frac{1}{\zeta}\left(\frac{\mu}{\sigma^2/2} - 1\right)$, we have

$$\varpi + k - \frac{2}{\zeta} = \frac{1}{\zeta}\left(\frac{\mu}{\sigma^2/2} - 1\right) + \frac{1}{\zeta}\sqrt{\left(\frac{\mu}{\sigma^2/2} - 1\right)^2 + 4\frac{r}{\sigma^2/2}} > 0$$

■

Lemma C.4

$$\begin{aligned}
\lim_{u\to 0} \frac{\theta(u)}{\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k} u^{-k-\varpi-1}} &= 1 \\
\lim_{u\to 0} \frac{\frac{\theta(u)}{\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k} u^{-k-\varpi-1}} - 1}{u^2} &= -\frac{1}{4(1+\varpi)}\left[\frac{2(\varpi-k)}{\varpi+2-k} + 1\right]
\end{aligned}$$

Proof. From (35), we have $\lim_{u\to 0} J_{\varpi,-k}(u) \frac{\int_0^u x^{-k-1} I_{\varpi}(x)dx}{u^{-k} I_{\varpi}(u)} = \frac{1}{\varpi-k}$ and from (**??**) $\lim_{u\to 0} \frac{I_{\varpi}(u)}{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}} = 1$, and $\varpi > |k|$. We thus have

$$\begin{aligned}
\lim_{u\to 0} \frac{\theta(u)}{\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k} u^{-k-\varpi-1}} &= \lim_{u\to 0} \frac{\frac{1}{u}\frac{\int_0^u t^{-k-1} I_{\varpi}(t)dt}{I_{\varpi}(u)^2}}{\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k} u^{-k-\varpi-1}} \\
&= (\varpi - k)\lim_{u\to 0} \frac{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}}{I_{\varpi}(u)} \frac{\int_0^u t^{-k-1} I_{\varpi}(t)\,dt}{u^{-k} I_{\varpi}(u)} \\
&= 1
\end{aligned}$$

Also from (36), we have $\lim_{\lambda\to 0}\frac{1}{\lambda^2}\left(J_{\varpi,-k}(\lambda)-\frac{1}{\varpi-k}\right)=-\frac{1}{4(\varpi+1)}\frac{2}{\varpi+2-k}\frac{1}{\varpi-k}$

$$\begin{aligned}
\lim_{u\to 0}\frac{\frac{\theta(u)}{\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}u^{-k-\varpi-1}}-1}{u^2} &= \lim_{u\to 0}\frac{(\varpi-k)\frac{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}}{I_{\varpi}(u)}\frac{\int_0^u t^{-k-1}I_{\varpi}(t)dt}{u^{-k}I_{\varpi}(u)}-1}{u^2}\\
&= \lim_{u\to 0}\frac{(\varpi-k)J_{\varpi,-k}(u)-\frac{I_{\varpi}(u)}{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}}}{\frac{I_{\varpi}(u)}{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}}u^2}\\
&= \lim_{u\to 0}\frac{(\varpi-k)\frac{J_{\varpi,-k}(u)-\frac{1}{\varpi-k}}{u^2}-\frac{\frac{I_{\varpi}(u)}{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}}-1}{u^2}}{\frac{I_{\varpi}(u)}{\frac{2^{-\varpi}u^{\varpi}}{\Gamma(\varpi+1)}}}\\
&= \frac{(\varpi-k)\left[-\frac{1}{4(\varpi+1)}\frac{2}{\varpi+2-k}\frac{1}{\varpi-k}\right]-\frac{1}{4(1+\varpi)}}{1}\\
&= -\frac{1}{4(1+\varpi)}\left[\frac{2}{\varpi+2-k}+1\right]
\end{aligned}$$

■

Lemma C.5

$$\lim_{\lambda\to 0}\int_0^{\lambda}\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt=\infty$$

Proof. From () and (36) There exists a $\delta>0$ and $C>0$ such that if $x\in(0,\delta]$,

$$\begin{aligned}
\left|\frac{I_{\varpi}(x)}{\frac{2^{-\varpi}x^{\varpi}}{\Gamma(\varpi+1)}}-1\right| &\leq Cx^2\\
\left|\frac{\theta(x)}{\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}x^{-k-\varpi-1}}-1\right| &\leq Cx^2
\end{aligned}$$

or

$$\begin{aligned}
\frac{2^{-\varpi}x^{\varpi}}{\Gamma(\varpi+1)}\left(1-Cx^2\right) &\leq I_{\varpi}(x)\leq\frac{2^{-\varpi}x^{\varpi}}{\Gamma(\varpi+1)}\left(1+Cx^2\right)\\
\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}x^{-k-\varpi-1}\left(1-Cx^2\right) &\leq \theta(x)\leq\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}x^{-k-\varpi-1}\left(1+Cx^2\right)
\end{aligned}$$

$$\left|\theta(x)-\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}x^{-k-\varpi-1}\right|\leq C\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}x^{2-k-\varpi-1}$$

The second implies that

$$\Theta(\lambda,\lambda s)=\int_{\lambda s}^{\lambda}\theta(u)\,du=\lambda\int_{s}^{1}\theta(\lambda r)\,dr$$

so that

$$\begin{aligned}
\left|\Theta(\lambda,\lambda s)-\lambda^{-k-\varpi}\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}\frac{s^{-\varpi+k}-1}{\varpi+k}\right| &= \left|\Theta(\lambda,\lambda s)-\lambda\int_{s}^{1}\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}(\lambda r)^{-k-\varpi-1}\,dr\right| \\
&= \left|\lambda\int_{s}^{1}\theta(\lambda r)\,dr-\lambda\int_{s}^{1}\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}(\lambda r)^{-k-\varpi-1}\,dr\right| \\
&= \left|\lambda\int_{s}^{1}\theta(\lambda r)-\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}(\lambda r)^{-k-\varpi-1}\,dr\right| \\
&\leq \lambda\int_{s}^{1}\left|\theta(\lambda r)-\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}(\lambda r)^{-k-\varpi-1}\right|dr \\
&\leq \lambda\int_{s}^{1}C\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}(\lambda r)^{2-k-\varpi-1}\,dr \\
&= \lambda^{2-k-\varpi}C\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}\frac{1-s^{2-k-\varpi}}{2-k-\varpi}
\end{aligned}$$

We then have

$$\begin{aligned}
\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}y(\lambda s,\lambda) &= \left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\left[\Theta(\lambda,\lambda s)\right] \\
&= \left\{\begin{array}{c}\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\,\lambda^{-k-\varpi}\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}\frac{s^{-\varpi+k}-1}{\varpi+k}\\ +\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\left[\Theta(\lambda,\lambda s)-\lambda^{-k-\varpi}\frac{\Gamma(\varpi+1)2^{\varpi}}{\varpi-k}\frac{s^{-\varpi+k}-1}{\varpi+k}\right]\end{array}\right\} \\
&= \left\{\frac{I_{\varpi}(\lambda s)}{\frac{2^{-\varpi}(\lambda s)^{\varpi}}{\Gamma(\varpi+1)}}\left(1-s^{\varpi+k}\right)+\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\left[\Theta(\lambda,\lambda s)-\lambda^{-k-\varpi}\frac{\Gamma(\varpi+1)\,2^{\varpi}}{\varpi-k}\frac{s^{-\varpi+}}{\varpi}\right.\right.
\end{aligned}$$

Then

$$\begin{aligned}
\left|1-\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}y(\lambda s,\lambda)-s^{\varpi+k}\right| &= \left|\left(1-\frac{I_{\varpi}(\lambda s)}{\frac{2^{-\varpi}(\lambda s)^{\varpi}}{\Gamma(\varpi+1)}}\right)\left(1-s^{\varpi+k}\right)-\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\left[\Theta(\lambda,\lambda s)-\lambda^{-k-}\right.\right. \\
&\leq \left|\left(1-\frac{I_{\varpi}(\lambda s)}{\frac{2^{-\varpi}(\lambda s)^{\varpi}}{\Gamma(\varpi+1)}}\right)\left(1-s^{\varpi+k}\right)\right|+\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\left|\Theta(\lambda,\lambda s)-\lambda^{-k-}\right. \\
&\leq \left|\left(1-\frac{I_{\varpi}(\lambda s)}{\frac{2^{-\varpi}(\lambda s)^{\varpi}}{\Gamma(\varpi+1)}}\right)\left(1-s^{\varpi+k}\right)\right|+\left(\varpi^{2}-k^{2}\right)(\lambda s)^{k}I_{\varpi}(\lambda s)\,\lambda^{2-k-\varpi}C\frac{\Gamma(\varpi+}{\varpi-} \\
&= \left|\left(1-\frac{I_{\varpi}(\lambda s)}{\frac{2^{-\varpi}(\lambda s)^{\varpi}}{\Gamma(\varpi+1)}}\right)\left(1-s^{\varpi+k}\right)\right|+\lambda^{2}\frac{I_{\varpi}(\lambda s)}{\frac{2^{-\varpi}(\lambda s)^{\varpi}}{\Gamma(\varpi+1)}}C(\varpi+k)\frac{s^{\varpi+k}-s^{2}}{2-k-\varpi}
\end{aligned}$$

Since $\left(1-s^{\varpi+k}\right)\leq 1$, the first term is bounded by $C\left(\lambda s\right)^{2}$. The second term is bounded by

$$\begin{aligned}
\lambda^{2}\frac{I_{\varpi}\left(\lambda s\right)}{\frac{2^{-\varpi}\left(\lambda s\right)^{\varpi}}{\Gamma(\varpi+1)}}C\left(\varpi+k\right)\frac{s^{\varpi+k}-s^{2}}{2-k-\varpi} &\leq \lambda^{2}\left(1+\left|\frac{I_{\varpi}\left(\lambda s\right)}{\frac{2^{-\varpi}\left(\lambda s\right)^{\varpi}}{\Gamma(\varpi+1)}}-1\right|\right)C\left(\varpi+k\right)\frac{s^{\varpi+k}-s^{2}}{2-k-\varpi}\\
&\leq \lambda^{2}\left(1+C\left(\lambda s\right)^{2}\right)C\left(\varpi+k\right)\frac{s^{\varpi+k}-s^{2}}{2-k-\varpi}\\
&\leq \lambda^{2}\left(1+C\left(\lambda s\right)^{2}\right)C\left(\varpi+k\right)\frac{s^{\varpi+k}+s^{2}}{\left|2-k-\varpi\right|}
\end{aligned}$$

Since $s\in\left|0,1\right|$, if we choose λ small enough so that $\lambda\leq\min\left\{\delta,C^{-1/2}\right\}$, then $1+C\left(\lambda s\right)^{2}\leq 2$, and thus the second term is bounded by

$$\lambda^{2}\frac{I_{\varpi}\left(\lambda s\right)}{\frac{2^{-\varpi}\left(\lambda s\right)^{\varpi}}{\Gamma(\varpi+1)}}C\left(\varpi+k\right)\frac{s^{\varpi+k}-s^{2}}{2-k-\varpi}\leq\lambda^{2}\frac{2C\left(\varpi+k\right)}{\left|2-k-\varpi\right|}\left(s^{\varpi+k}+s^{2}\right)$$

Together, we have that if $\lambda\leq\min\left\{\delta,C^{-1/2}\right\}$, then

$$\begin{aligned}
\left|1-\left(\varpi^{2}-k^{2}\right)\left(\lambda s\right)^{k}y\left(\lambda s,\lambda\right)-s^{\varpi+k}\right| &\leq C\left(\lambda s\right)^{2}+\lambda^{2}\frac{2C\left(\varpi+k\right)}{\left|2-k-\varpi\right|}\left(s^{\varpi+k}+s^{2}\right)\\
&\leq \lambda^{2}C\left(1+\frac{2\left(\varpi+k\right)}{\left|2-k-\varpi\right|}\right)\left(s^{\varpi+k}+s^{2}\right)
\end{aligned}$$

With this, we have that

$$\begin{aligned}
\int_{0}^{\lambda}\left[1-\left(\varpi^{2}-k^{2}\right)t^{k}y\left(t,\lambda\right)\right]t^{-\frac{2}{\zeta}-1}dt &= \lambda^{-\frac{2}{\zeta}}\int_{0}^{1}\left[1-\left(\varpi^{2}-k^{2}\right)\left(\lambda s\right)^{k}y\left(\lambda s,\lambda\right)\right]s^{-\frac{2}{\zeta}-1}ds\\
&= \lambda^{-\frac{2}{\zeta}}\left\{\int_{0}^{1}s^{\varpi+k}s^{-\frac{2}{\zeta}-1}ds+\int_{0}^{1}\left[1-\left(\varpi^{2}-k^{2}\right)\left(\lambda s\right)^{k}y\left(\lambda s,\lambda\right)-s^{\varpi+k}\right]s^{-\frac{2}{\zeta}-1}\right.\\
&\geq \lambda^{-\frac{2}{\zeta}}\left\{\int_{0}^{1}s^{\varpi+k}s^{-\frac{2}{\zeta}-1}ds-\int_{0}^{1}\lambda^{2}C\left(1+\frac{2\left(\varpi+k\right)}{\left|2-k-\varpi\right|}\right)\left(s^{\varpi+k}+s^{2}\right)s^{-\frac{2}{\zeta}-1}ds\right.\\
&= \lambda^{-\frac{2}{\zeta}}\frac{1}{\varpi+k-\frac{2}{\zeta}}-\lambda^{2-\frac{2}{\zeta}}C\left(1+\frac{2\left(\varpi+k\right)}{\left|2-k-\varpi\right|}\right)\left[\frac{1}{\varpi+k-\frac{2}{\zeta}}-\frac{1}{2-\frac{2}{\zeta}}\right]
\end{aligned}$$

As $\lambda\rightarrow 0$, the first term diverges and the second term goes to zero. Therefore

$$\lim_{\lambda\rightarrow 0}\int_{0}^{\lambda}\left[1-\left(\varpi^{2}-k^{2}\right)t^{k}y\left(t,\lambda\right)\right]t^{-\frac{2}{\zeta}-1}dt=\infty$$

■

Lemma C.6 $\lim_{\lambda\rightarrow 0}H\left(\lambda\right)=\frac{1}{\varpi+k-\frac{2}{\zeta}}$

Proof. Using L'Hopital's rule gives

$$\begin{aligned}
\lim_{\lambda\to 0} H(\lambda) &= \lim_{\lambda\to 0} \frac{\int_0^\lambda \left[1-\left(\varpi^2-k^2\right) t^k y(t,\lambda)\right] t^{-\frac{2}{\zeta}-1} dt}{\lambda^{-\frac{2}{\zeta}}} \\
&= \lim_{\lambda\to 0} \frac{\left[1-\left(\varpi^2-k^2\right)\lambda^k y(\lambda,\lambda)\right]\lambda^{-\frac{2}{\zeta}-1} + \int_0^\lambda \left[-\left(\varpi^2-k^2\right) t^k \frac{dy(t,\lambda)}{d\lambda}\right] t^{-\frac{2}{\zeta}-1} dt}{-\frac{2}{\zeta}\lambda^{-\frac{2}{\zeta}-1}}
\end{aligned}$$

Using $y(\lambda,\lambda)=0$ and $\frac{dy(t,\lambda)}{d\lambda} = I_\varpi(t)\,\theta(\lambda)$, this is

$$\begin{aligned}
\lim_{\lambda\to 0} H(\lambda) &= \lim_{\lambda\to 0} -\frac{1}{\frac{2}{\zeta}} - \left(\varpi^2-k^2\right)\frac{\int_0^\lambda t^k I_\varpi(t)\,\theta(\lambda)\, t^{-\frac{2}{\zeta}-1} dt}{-\frac{2}{\zeta}\lambda^{-\frac{2}{\zeta}-1}} \\
&= \lim_{\lambda\to 0} -\frac{1}{\frac{2}{\zeta}} - \left(\varpi^2-k^2\right)\frac{\frac{\int_0^\lambda t^{-k-1} I_\varpi(t) dt}{\lambda I_\varpi(\lambda)^2}\int_0^\lambda t^{k-\frac{2}{\zeta}-1} I_\varpi(t)\, dt}{-\frac{2}{\zeta}\lambda^{-\frac{2}{\zeta}-1}} \\
&= \lim_{\lambda\to 0} -\frac{1}{\frac{2}{\zeta}} + \left(\varpi^2-k^2\right)\frac{1}{\frac{2}{\zeta}}\frac{\int_0^\lambda t^{-k-1} I_\varpi(t)\, dt}{\lambda^{-k} I_\varpi(\lambda)}\frac{\int_0^\lambda t^{k-\frac{2}{\zeta}-1} I_\varpi(t)\, dt}{\lambda^{k-\frac{2}{\zeta}} I_\varpi(\lambda)} \\
&= \lim_{\lambda\to 0} -\frac{1}{\frac{2}{\zeta}} + \left(\varpi^2-k^2\right)\frac{1}{\frac{2}{\zeta}} J_{\varpi,-k}(\lambda)\, J_{\varpi,k-\frac{2}{\zeta}}(\lambda)
\end{aligned}$$

Finally, using lemma (35) these limits are

$$\begin{aligned}
\lim_{\lambda\to 0} H(\lambda) &= -\frac{1}{\frac{2}{\zeta}} + \left(\varpi^2-k^2\right)\frac{1}{\frac{2}{\zeta}}\frac{1}{\varpi-k}\frac{1}{\varpi+k-\frac{2}{\zeta}} \\
&= -\frac{1}{\frac{2}{\zeta}} + \frac{1}{\frac{2}{\zeta}}\frac{\varpi+k}{\varpi+k-\frac{2}{\zeta}} \\
&= \frac{1}{\varpi+k-\frac{2}{\zeta}}
\end{aligned}$$

■

Lemma C.7

$$\lim_{\lambda\to 0} \frac{H(\lambda) - \frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2} = \frac{\zeta}{\zeta-1}\frac{1}{4(1+\varpi)}\frac{\varpi+k}{\varpi+k-\frac{2}{\zeta}}\left[\frac{1}{2+\varpi-k} + \frac{1}{2+\varpi+k-\frac{2}{\zeta}}\right] > 0$$

Proof.

$$
\begin{aligned}
\lim_{\lambda\to 0}\frac{H(\lambda)-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2} &= \lim_{\lambda\to 0}\frac{\lambda^{\frac{2}{\zeta}}\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2}\\
&= \lim_{\lambda\to 0}\frac{\frac{2}{\zeta}\lambda^{\frac{2}{\zeta}-1}\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt+\lambda^{\frac{2}{\zeta}}\frac{d}{d\lambda}\left[\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}d\right.}{2\lambda}\\
&= \lim_{\lambda\to 0}\frac{1}{\zeta}\frac{H(\lambda)}{\lambda^2}+\frac{\lambda^{\frac{2}{\zeta}}\frac{d}{d\lambda}\left[\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt\right]}{2\lambda}\\
&= \lim_{\lambda\to 0}\frac{1}{\zeta}\frac{H(\lambda)-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2}+\frac{1}{\zeta}\frac{\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2}+\frac{\lambda^{\frac{2}{\zeta}}\frac{d}{d\lambda}\left[\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt\right]}{2\lambda}
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
\lim_{\lambda\to 0}\frac{H(\lambda)-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2} &= \frac{1}{1-\frac{1}{\zeta}}\lim_{\lambda\to 0}\frac{\frac{\frac{2}{\zeta}}{\varpi+k-\frac{2}{\zeta}}}{2\lambda^2}+\frac{\lambda^{\frac{2}{\zeta}}\frac{d}{d\lambda}\left[\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt\right]}{2\lambda}\\
&= \frac{1}{1-\frac{1}{\zeta}}\lim_{\lambda\to 0}\frac{\frac{\varpi+k}{\varpi+k-\frac{2}{\zeta}}-1}{2\lambda^2}+\frac{\lambda^{\frac{2}{\zeta}}\left\{\left[1-\left(\varpi^2-k^2\right)\lambda^k y(\lambda,\lambda)\right]\lambda^{-\frac{2}{\zeta}-1}+\int_0^\lambda\left[-\left(\varpi^2-k^2\right)t^k\frac{dy(t,\lambda)}{d\lambda}\right]t\right.}{2\lambda}\\
&= \frac{1}{1-\frac{1}{\zeta}}\lim_{\lambda\to 0}\frac{\frac{1}{\varpi-k}\frac{\varpi^2-k^2}{\varpi+k-\frac{2}{\zeta}}-1}{2\lambda^2}+\frac{\lambda^{\frac{2}{\zeta}}\left\{\lambda^{-\frac{2}{\zeta}-1}-\left(\varpi^2-k^2\right)\int_0^\lambda t^k I_\varpi(t)\,\theta(\lambda)\,t^{-\frac{2}{\zeta}-1}dt\right\}}{2\lambda}\\
&= \frac{1}{1-\frac{1}{\zeta}}\lim_{\lambda\to 0}\frac{\frac{1}{\varpi-k}\frac{\varpi^2-k^2}{\varpi+k-\frac{2}{\zeta}}-1}{2\lambda^2}+\frac{\frac{1}{\lambda}-\left(\varpi^2-k^2\right)\frac{1}{\lambda}\frac{\int_0^\lambda I_\varpi(t)t^{-k-1}dt}{\lambda^{-k}I_\varpi(\lambda)}\frac{\int_0^\lambda I_\varpi(t)t^{k-\frac{2}{\zeta}-1}dt}{\lambda^{k-\frac{2}{\zeta}}I_\varpi(\lambda)}}{2\lambda}\\
&= \frac{1}{1-\frac{1}{\zeta}}\lim_{\lambda\to 0}\frac{\frac{1}{\varpi-k}\frac{\varpi^2-k^2}{\varpi+k-\frac{2}{\zeta}}-1}{2\lambda^2}+\frac{1-\left(\varpi^2-k^2\right)J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda)}{2\lambda^2}\\
&= \frac{\varpi^2-k^2}{1-\frac{1}{\zeta}}\frac{1}{2}\lim_{\lambda\to 0}\frac{\frac{1}{\varpi-k}\frac{1}{\varpi+k-\frac{2}{\zeta}}-J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda)}{\lambda^2}
\end{aligned}
$$

Rearranging and using the lemma $\lim_{\lambda\to 0}\frac{1}{\lambda^2}\left(J_{\varpi,c}(\lambda)-\frac{1}{\varpi+c}\right)=-\frac{1}{4(\varpi+1)}\frac{2}{\varpi+2+c}\frac{1}{c+\varpi}$, the inner limit can be expressed as

$$\begin{aligned}
\lim_{\lambda\to0}\frac{\frac{1}{\varpi-k}\frac{1}{\varpi+k-\frac{2}{\zeta}}-J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda)}{\lambda^2} &= \lim_{\lambda\to0}\frac{\frac{1}{\varpi-k}-J_{\varpi,-k}(\lambda)}{\lambda^2}J_{\varpi,k-\frac{2}{\zeta}}(\lambda)+\frac{1}{\varpi-k}\frac{\frac{1}{\varpi+k-\frac{2}{\zeta}}-J_{\varpi,k-\frac{2}{\zeta}}(\lambda)}{\lambda^2}\\
&= \frac{1}{4(1+\varpi)}\frac{2}{2+\varpi-k}\frac{1}{\varpi-k}\frac{1}{\varpi+k-\frac{2}{\zeta}}+\frac{1}{4(1+\varpi)}\frac{1}{\varpi-k}\frac{2}{2+\varpi+k-\frac{2}{\zeta}}\frac{}{\varpi}\\
&= \frac{1}{2(1+\varpi)}\frac{1}{\varpi-k}\frac{1}{\varpi+k-\frac{2}{\zeta}}\left[\frac{1}{2+\varpi-k}+\frac{1}{2+\varpi+k-\frac{2}{\zeta}}\right]\\
&> 0
\end{aligned}$$

■

Lemma C.8 *$H'(0)=0$, $H''(0)>0$.*

Proof. The definition of H directly gives

$$H(0)=\lim_{\lambda\to0}H(\lambda)=\frac{1}{\varpi+k-\frac{2}{\zeta}}>0$$

By definition,

$$H'(0)=\lim_{\lambda\to0}\frac{H(\lambda)-H(0)}{\lambda}=\lim_{\lambda\to0}\frac{H(\lambda)-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda}=\lim_{\lambda\to0}\lambda\lim_{\lambda\to0}\frac{H(\lambda)-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2}=0$$

Similarly, $H''(0)=2\lim_{\lambda\to0}\frac{H(\lambda)-\frac{1}{\varpi+k-\frac{2}{\zeta}}}{\lambda^2}>0$. ■

Claim C.9 *H is a strictly increasing function.*

Proof. We know that $H'(0)=0$ and $H''(0)>0$. This means that there is an $\varepsilon>0$ such that $H'(0)>0$ for all $\lambda\in(0,\varepsilon]$. Towards a contradiction, suppose that there exists a $\lambda_0=\inf_\lambda\{\lambda|\lambda\geq\varepsilon \text{ and } H'(0)=0\}$. We will show that it must be that $H''(\lambda_0)>0$, which contradicts the idea that λ_0 such an infimum exists. Rearranging the definition of H and differentiating gives and using $\frac{dy(t,\lambda)}{d\lambda}=I_\varpi(t)\,\theta(\lambda)$

$$\lambda^{-2/\zeta}H(\lambda)=\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt$$

$$
\begin{aligned}
-\frac{2}{\zeta}\lambda^{-\frac{2}{\zeta}-1}H(\lambda)+\lambda^{-2/\zeta}H'(\lambda) &= \lambda^{-\frac{2}{\zeta}-1}-\left(\varpi^2-k^2\right)\int_0^\lambda t^k\frac{dy(t,\lambda)}{d\lambda}t^{-\frac{2}{\zeta}-1}dt \\
&= \lambda^{-\frac{2}{\zeta}-1}-\left(\varpi^2-k^2\right)\int_0^\lambda t^k I_\varpi(t)\,\theta(\lambda)\,t^{-\frac{2}{\zeta}-1}dt \\
&= \lambda^{-\frac{2}{\zeta}-1}-\left(\varpi^2-k^2\right)\theta(\lambda)\int_0^\lambda I_\varpi(t)\,t^{k-\frac{2}{\zeta}-1}dt \\
&= \lambda^{-\frac{2}{\zeta}-1}-\left(\varpi^2-k^2\right)\frac{\int_0^\lambda u^{-k-1}I_\varpi(u)\,du}{\lambda I_\varpi(\lambda)^2}\int_0^\lambda I_\varpi(t)\,t^{k-\frac{2}{\zeta}-1}dt \\
&= \lambda^{-\frac{2}{\zeta}-1}-\left(\varpi^2-k^2\right)\lambda^{-\frac{2}{\zeta}-1}J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda)
\end{aligned}
$$

Dividing through by $\lambda^{-\frac{2}{\zeta}-1}$ and differentiating once more gives

$$
\begin{aligned}
-\frac{2}{\zeta}H(\lambda)+\lambda H'(\lambda) &= 1-\left(\varpi^2-k^2\right)J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda) \\
-\frac{2}{\zeta}H'(\lambda)+H'(\lambda)+\lambda H''(\lambda) &\equiv -\left(\varpi^2-k^2\right)\frac{d}{d\lambda}\left[J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda)\right]
\end{aligned}
$$

Evaluating this at λ_0, the gives

$$
\lambda_0 H''(\lambda_0)\equiv-\left(\varpi^2-k^2\right)\frac{d}{d\lambda}\left[J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda)\right]\Bigg|_{\lambda=\lambda_0}
$$

Since $J_{\varpi,-k}(\lambda)$ and $J_{\varpi,k-\frac{2}{\zeta}}(\lambda)$ are each positive and decreasing, $\lambda_0 H''(\lambda_0)>0$, and hence $\lambda_0 H''(\lambda_0)>0$, which gives the contradiction. ■

Lemma C.10 $\lambda^{-\frac{2}{\zeta}}\left[1+\frac{2}{\zeta}H(\lambda)\right]$ *is decreasing in* λ.

Proof. Using the definition of H, we have

$$
\lambda^{-\frac{2}{\zeta}}\left[1+\frac{2}{\zeta}H(\lambda)\right]=\lambda^{-\frac{2}{\zeta}}+\frac{2}{\zeta}\int_0^\lambda\left[1-\left(\varpi^2-k^2\right)t^k y(t,\lambda)\right]t^{-\frac{2}{\zeta}-1}dt
$$

Differentiating and using $y(\lambda,\lambda)=0$ and $\frac{dy(t,\lambda)}{d\lambda}=I_\varpi(t)\,\theta(\lambda)$, we have

$$\begin{aligned}
\frac{d}{d\lambda}\left\{\lambda^{-\frac{2}{\zeta}}\left[1+\frac{2}{\zeta}H(\lambda)\right]\right\} &= -\frac{2}{\zeta}\lambda^{-\frac{2}{\zeta}-1}+\frac{2}{\zeta}\left[1-\left(\varpi^2-k^2\right)\lambda^k y(\lambda,\lambda)\right]\lambda^{-\frac{2}{\zeta}-1}+\frac{2}{\zeta}\int_0^\lambda\left[-\left(\varpi^2-k^2\right)t^k\frac{dy(t,\lambda)}{d\lambda}\right]t^{-\frac{2}{\zeta}-1}dt \\
&= -\left(\varpi^2-k^2\right)\frac{2}{\zeta}\int_0^\lambda t^k I_\varpi(t)\,\theta(\lambda)\,t^{-\frac{2}{\zeta}-1}dt \\
&= -\left(\varpi^2-k^2\right)\frac{2}{\zeta}\frac{\int_0^\lambda u^{-k-1}I_\varpi(u)\,du}{uI_\varpi(u)^2}\int_0^\lambda t^k I_\varpi(t)\,t^{-\frac{2}{\zeta}-1}dt \\
&= -\left(\varpi^2-k^2\right)\frac{2}{\zeta}J_{\varpi,-k}(\lambda)\,J_{\varpi,k-\frac{2}{\zeta}}(\lambda) \\
&< 0
\end{aligned}$$

$$\lim_{\lambda\to\infty}\lambda^{-\frac{2}{\zeta}}\left[1+\frac{2}{\zeta}H(\lambda)\right]=$$

■

Proposition C.11 *Suppose that $\frac{h(q,a)}{\Phi_a}$ is increasing in a. Then $\frac{\eta_a}{F_a'(a)}$ and b_a are increasing in a.*

Proof. By definition, we have

$$b_a\equiv\frac{M}{J}\underline{x}_a^{-\zeta}\frac{\int h(q,a)^\zeta s(q)\,dF_q(q)}{\int s(q)\,dF_q(q)}$$

easier to work with $\lambda_a\equiv\left(\frac{(1-\beta)b_a}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}\right)^{1/2}$, in which case this is

$$\frac{1}{1-\beta}\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2\lambda_a^2\equiv\frac{M}{J}\underline{x}_a^{-\zeta}\frac{\int h(q,a)^\zeta s(q)\,dF_q(q)}{\int s(q)\,dF_q(q)}$$

Using $\underline{x}_a=\frac{\frac{r-\mu}{r}\Phi_a w^{\frac{1}{\alpha}}}{1+\frac{2}{\zeta}H(\lambda_a)}$, this is

$$\frac{1}{1-\beta}\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2\lambda_a^2\equiv\frac{M}{J}\left(\frac{\frac{r-\mu}{r}\Phi_a w^{\frac{1}{\alpha}}}{1+\frac{2}{\zeta}H(\lambda_a)}\right)^{-\zeta}\frac{\int h(q,a)^\zeta s(q)\,dF_q(q)}{\int s(q)\,dF_q(q)}$$

We can rearrange this as

$$\frac{1}{1-\beta}\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2\lambda_a^2\left[1+\frac{2}{\zeta}H(\lambda_a)\right]^{-\zeta}\equiv\frac{M}{J}\left(\frac{r-\mu}{r}w^{\frac{1}{\alpha}}\right)^{-\zeta}\frac{\int\left[\frac{h(q,a)}{\Phi_a}\right]^\zeta s(q)\,dF_q(q)}{\int s(q)\,dF_q(q)}$$

The RHS is increasing in a because $\frac{h(q,a)}{\Phi_a}$ is increasing for each a. The LHS is increasing in λ_a because, as we showed in Lemma XX, $\lambda^{-\frac{2}{\zeta}}\left[1+\frac{2}{\zeta}H(\lambda_a)\right]$ is decreasign in λ. Together, these imply that λ_a (and hence b_a) is increasing in a.

Similarly, η_a is defined as

$$
\begin{aligned}
\frac{\eta_a}{F_a'(a)} &\equiv \Upsilon \underline{x}_a^{-\zeta} \int h(q,a)^{\zeta} n(q)\, dF_q(q) \\
&= \Upsilon \left(\frac{\frac{r-\mu}{r} \Phi_a w^{\frac{1}{\alpha}}}{1+\frac{2}{\zeta} H(\lambda_a)} \right)^{-\zeta} \int h(q,a)^{\zeta} n(q)\, dF_q(q) \\
&= \Upsilon \left(\frac{r-\mu}{r} w^{\frac{1}{\alpha}} \right)^{-\zeta} \left[1 + \frac{2}{\zeta} H(\lambda_a) \right]^{\zeta} \int \left[\frac{h(q,a)}{\Phi_a} \right]^{\zeta} n(q)\, dF_q(q)
\end{aligned}
$$

The RHS is increasing in a because $\frac{h(q,a)}{\Phi_a}$ is increasing for each a, $1+\frac{2}{\zeta}H(\lambda)$ is increasing in λ, and λ_a is increasing in a. ■

Claim C.12 *Define the function* $T(\lambda)$ *as*

$$
T(\lambda) = \int_0^{\lambda} y(t,\lambda)\, t^{k+2-\frac{2}{\zeta}-1} dt
$$

Then its derivative is

$$
T'(\lambda) = \lambda^{2-\frac{2}{\zeta}-1} J_{\varpi,-k}(\lambda)\, J_{\varpi,k+2-\frac{2}{\zeta}}(\lambda)
$$

Then

$$
\int_{\underline{x}_a}^{\infty} v_a(x)\, \zeta x^{-\zeta-1} dx = \frac{w^{-\frac{1}{\alpha}}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3} \underline{x}_a^{1-\zeta} \lambda_a^{\frac{2}{\zeta}-2} T(\lambda_a)
$$

In addition, $\lim_{a\to\infty} T(\lambda_a) < \infty$

$$
\lim_{a\to\infty} \frac{\int_{\underline{x}_a}^{\infty} v_a(x)\, \zeta x^{-\zeta-1} dx}{\left(w^{\frac{1}{\alpha}} \Phi_a\right)^{1-\zeta}} = \frac{w^{-\frac{1}{\alpha}} \left(\frac{r-\mu}{r}\right)^{1-\zeta}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3} \lim_{a\to\infty} \frac{\lambda_a^{\frac{2}{\zeta}-2} T(\lambda_a)}{\left(1+\frac{2}{\zeta} H(\lambda_a)\right)^{1-\zeta}}
$$

Proof. Integrating by parts, making the change of variables $x = \underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}$, and using $y(t,\lambda_a) \equiv w^{\frac{1}{\alpha}} \frac{\sigma^2}{2} \left(\frac{\zeta}{2}\right)^2 t^{-k} v_a'\left(\underline{x}_a \lambda_a^{\frac{2}{\zeta}}\right.$

$$
\begin{aligned}
\int_{\underline{x}_a}^{\infty} v_a(x)\, \zeta x^{-\zeta-1} dx &= \int_{\underline{x}_a}^{\infty} v_a'(x)\, x^{-\zeta} dx \\
&= \underline{x}_a^{1-\zeta} \lambda_a^{\frac{2}{\zeta}-2} \int_0^{\lambda_a} v_a'\left(\underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}\right) \frac{2}{\zeta} t^{2-\frac{2}{\zeta}-1} dt \\
&= \frac{w^{-\frac{1}{\alpha}} \underline{x}_a^{1-\zeta} \lambda_a^{\frac{2}{\zeta}-2}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2} \frac{2}{\zeta} \int_0^{\lambda_a} y(t,\lambda_a)\, t^{k+2-\frac{2}{\zeta}-1} dt
\end{aligned}
$$

Since $y(\lambda,\lambda)=0$ and $\frac{dy(t,\lambda)}{d\lambda}=I_{\varpi}(t)\,\theta(\lambda)$, differentiating the integral gives

$$\begin{aligned}
T'(\lambda) &= \frac{d}{d\lambda}\left\{\int_0^{\lambda} y(t,\lambda)\,t^{2+k}t^{-\frac{2}{\zeta}-1}dt\right\} = \int_0^{\lambda}\frac{dy(t,\lambda)}{d\lambda}t^{2+k}t^{-\frac{2}{\zeta}-1}dt \\
&= \int_0^{\lambda} I_{\varpi}(t)\,\theta(\lambda)\,t^{2+k}t^{-\frac{2}{\zeta}-1}dt \\
&= \theta(\lambda)\int_0^{\lambda} I_{\varpi}(t)\,t^{2+k}t^{-\frac{2}{\zeta}-1}dt \\
&= \lambda^{2-\frac{2}{\zeta}-1}\frac{\int_0^{\lambda}u^{-k-1}I_{\varpi}(u)\,du}{\lambda^{-k}I_{\varpi}(\lambda)}\frac{\int_0^{\lambda}I_{\varpi}(t)\,t^{2+k}t^{-\frac{2}{\zeta}-1}dt}{\lambda^{k+2-\frac{2}{\zeta}}I_{\varpi}(\lambda)} \\
&= \lambda^{2-\frac{2}{\zeta}-1}J_{\varpi,-k}(\lambda)\,J_{\varpi,k+2-\frac{2}{\zeta}}(\lambda)
\end{aligned}$$

Finally, note that

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{T'(\lambda)}{\lambda^{-\frac{2}{\zeta}-1}} &= \frac{\frac{1}{\lambda I_{\varpi}(\lambda)^2}\int_0^{\lambda}\tilde{t}^{-k-1}I_{\varpi}\left(\tilde{t}\right)d\tilde{t}\int_0^{\lambda}I_{\varpi}(t)\,t^{2+k}t^{-\frac{2}{\zeta}-1}dt}{\lambda^{-\frac{2}{\zeta}-1}} \\
&= \frac{\int_0^{\lambda}\tilde{t}^{-k-1}I_{\varpi}\left(\tilde{t}\right)d\tilde{t}}{I_{\varpi}(\lambda)\,\lambda^{-k-1}}\frac{\int_0^{\lambda}I_{\varpi}(t)\,t^{2+k}t^{-\frac{2}{\zeta}-1}dt}{I_{\varpi}(\lambda)\,\lambda^{2+k-\frac{2}{\zeta}-1}} \\
&= 1
\end{aligned}$$

Therefore $T(\lambda)$ is increasing and $\lim_{\lambda\to\infty}T(\lambda)<\infty$. Therefore $\lim_{a\to\infty}T(\lambda_a)<\infty$. In addition, using $\underline{x}_a=\frac{\frac{r-\mu}{r}\Phi_a w^{\frac{1}{\alpha}}}{1+\frac{2}{\zeta}H(\lambda_a)}$, we have

$$\begin{aligned}
\frac{\int_{\underline{x}_a}^{\infty}v_a(x)\,\zeta x^{-\zeta-1}dx}{\left(w^{\frac{1}{\alpha}}\Phi_a\right)^{1-\zeta}} &= \frac{w^{-\frac{1}{\alpha}}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3}\left(\frac{\underline{x}_a}{w^{\frac{1}{\alpha}}\Phi_a}\right)^{1-\zeta}\lambda_a^{\frac{2}{\zeta}-2}T(\lambda_a) \\
&= \frac{w^{-\frac{1}{\alpha}}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3}\left(\frac{\frac{r-\mu}{r}}{1+\frac{2}{\zeta}H(\lambda_a)}\right)^{1-\zeta}\lambda_a^{\frac{2}{\zeta}-2}T(\lambda_a) \\
&= \frac{w^{-\frac{1}{\alpha}}\left(\frac{r-\mu}{r}\right)^{1-\zeta}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3}\frac{\lambda_a^{\frac{2}{\zeta}-2}T(\lambda_a)}{\left(1+\frac{2}{\zeta}H(\lambda_a)\right)^{1-\zeta}}
\end{aligned}$$

Taking the limit gives the result. ■

D Stationary Distribution

D.1 Derivation of the Kolmogorov Forward Equation

Proposition D.1 *Among plants of type a, the stationary measure of plants with x satisfies (per firm) satisfies*

$$0=-g_L G_a'(x)-\frac{d}{dx}\left[\mu x G_a'(x)\right]+\frac{d^2}{dx^2}\left[\frac{\sigma^2 x^2}{2}G_a'(x)\right]+N_a'(x)-\underbrace{G_a'(x)\,B_a x^{-\zeta}}_{\text{plants with } x,a \text{ that are acquired}}+\underbrace{\zeta x^{-\zeta-1}B_a G_a(x)}_{\text{grafting of plants that become } x,a}$$

with initial and terminal conditions.

$$\begin{aligned}
G_a(\underline{x}_a) &= 0\\
G_a'(\underline{x}_a) &= 0\\
N_a(\infty) &= \frac{\sigma_x^2}{2}\underline{x}_a^2 G_a''(\underline{x}_a)
\end{aligned}$$

Proof. We start Let $\tilde{G}_a(x)$ be the measure of plants of type a with plant productivity weakly less than x.

$$\begin{aligned}
\frac{d}{dt}\tilde{G}_a'(x,t) &= -\frac{d}{dx}\left[\mu_x x\tilde{G}_a'(x,t)\right]+\frac{d^2}{dx^2}\left[\frac{\sigma_x^2}{2}x^2\tilde{G}_a'(x,t)\right]+\tilde{N}_a'(x,t)\\
&\quad -\underbrace{\tilde{G}_a'(x,t)\frac{m(J_t,S_t)}{J_t}\int h(q,a)^{\zeta}\frac{s(q,t)}{S}d\tilde{F}_q(q)\int_0^{\infty}1\{u>a\}\zeta u^{-\zeta-1}du}_{\text{plants with } x,a \text{ that are acquired}}\\
&\quad +\underbrace{\frac{m(J_t,S_t)}{S_t}\int s(q,t)h(q,a)^{\zeta}\zeta x^{-\zeta-1}d\tilde{F}_q(q)\frac{1}{J_t}\int 1\{x>u\}d\tilde{G}_a(u,t)}_{\text{grafting of plants that become } x,a}
\end{aligned}$$

Using $B_{a,t}=\frac{m(J_t,S_t)}{J_t}\int h(q,a)^{\zeta}\frac{s(q,t)}{S_t}d\tilde{F}_q(q)$, this can be simplified to

$$\begin{aligned}
\frac{d}{dt}\tilde{G}_a'(x,t) &= -\frac{d}{dx}\left[\mu_x x\tilde{G}_a'(x,t)\right]+\frac{d^2}{dx^2}\left[\frac{\sigma_x^2}{2}x^2\tilde{G}_a'(x,t)\right]+\tilde{N}_a'(x,t)\\
&\quad -\tilde{G}_a'(x,t)B_{a,t}\int_x^{\infty}\zeta u^{-\zeta-1}du+\zeta x^{-\zeta-1}B_{a,t}\int_{\underline{x}_{a,t}}^{x}d\tilde{G}_a(u,t)
\end{aligned}$$

Integrating and using $\tilde{G}_a\left(\underline{x}_{a,t},t\right)=0$ gives

$$\begin{aligned}
\frac{d}{dt}\tilde{G}_a'(x,t) &= -\frac{d}{dx}\left[\mu_x x\tilde{G}_a'(x,t)\right]+\frac{d^2}{dx^2}\left[\frac{\sigma_x^2}{2}x^2\tilde{G}_a'(x,t)\right]+\tilde{N}_a'(x,t)\\
&\quad -\tilde{G}_a'(x,t)B_{a,t}x^{-\zeta}+\zeta x^{-\zeta-1}B_{a,t}\tilde{G}_a(x,t)
\end{aligned}$$

Since $G_a(x,t) = \frac{\tilde{G}_a(x,t)}{J_t}$, we have

$$\frac{dG_a(x,t)}{dt} = -\frac{\tilde{G}_a(x,t)}{J_t}\frac{\dot{J}_t}{J_t} + \frac{\frac{d\tilde{G}_a(x,t)}{dt}}{J_t}$$

Therefore

$$\begin{aligned}\frac{d}{dt}G'_a(x,t) &= -G'_a(x,t)\frac{\dot{J}_t}{J_t} - \frac{d}{dx}\left[\mu_x x G'_a(x,t)\right] + \frac{d^2}{dx^2}\left[\frac{\sigma_x^2}{2}x^2 G'_a(x,t)\right] + N'_a(x,t)\\ &\quad -G'_a(x,t)\,B_{a,t}x^{-\zeta} + \zeta x^{-\zeta-1}B_{a,t}G_a(x,t)\end{aligned}$$

In steady state, $\frac{d}{dt}G'_a(x) = 0$ and $\frac{\dot{J}}{J} = g_L$. We can simplify this to get

$$\begin{aligned}0 &= -g_L G'_a(x,t) - \frac{d}{dx}\left[\mu_x x G'_a(x,t)\right] + \frac{d^2}{dx^2}\left[\frac{\sigma_x^2}{2}x^2 G'_a(x,t)\right] + N'_a(x,t)\\ &\quad -G'_a(x,t)\,B_{a,t}x^{-\zeta} + \zeta x^{-\zeta-1}B_{a,t}G_a(x,t)\end{aligned}$$

There are several terminal conditions.

$$\begin{aligned}G_a(\underline{x}_a) &= 0\\ G'_a(\underline{x}_a) &= 0\\ N_a(\infty) &= \frac{\sigma_x^2}{2}\underline{x}_a^2 G''_a(\underline{x}_a)\end{aligned}$$

For a heuristic derivation of the last two conditions, we can use a discrete approximation to the Brownian motion. Let $y = \log x$. In a time interval Δ, y goes up by $h(\Delta)$ with probability $p(\Delta)$ and down by $h(\Delta)$ with probability $1 - p(\Delta)$. We parameterize h and Δ so that

$$\begin{aligned}\frac{(2p-1)\,h}{\Delta} &= \mu\\ \frac{h^2}{\Delta} &= \sigma^2\end{aligned}$$

Let $Q(y)$ be the measure of y. so that $Q(y,t) = \tilde{G}_a(e^y,t)$. At the exit threshold $\underline{y} = \log \underline{x}_a$, the density of y just above the threshold satisfies

$$Q'\left(\underline{y}+h, t+\Delta\right) = (1-p)\,\Gamma'\left(\underline{y}+2h, t\right) + \Delta N'\left(\underline{y}\right)$$

Taking a limit as $\Delta \to 0$ gives $Q'\left(\underline{y},t\right) = \frac{1}{2}Q'\left(\underline{y},t\right)$, or $Q'\left(\underline{y},t\right) = 0$. This implies $0 = Q'\left(\underline{y},t\right) = e^{\underline{y}}\tilde{G}'_a(e^{\underline{y}},t) = \underline{x}(a)\,\tilde{G}'_a(\underline{x}_a,t)$, or simply $\tilde{G}'_a(\underline{x}_a) = 0$. This gives $G'(\underline{x}_a,t) = 0$.

Second, the change in the measure of plants is equal to the difference between the flow of entering plants and the flow of exiting plants. The flow of entering plants is $\tilde{N}(\infty,t)$. To get at the flow of exiting plants, we again use the discrete approximation. The measure that enter at $t+\Delta$ is $t+\Delta$ is $(1-p)\,Q'\left(\underline{y}+h,t\right)h$. Using a first order Taylor explansion around $\underline{y}$ gives $Q'\left(\underline{y}+h\right)=Q'\left(\underline{y}\right)+Q''\left(\underline{y}\right)h+O\left(h^2\right)$. Plugging this in, dividing by Δ gives the outflow per unit of time:

$$\frac{(1-p)\left[Q'\left(\underline{y}\right)+Q''\left(\underline{y}\right)h+O\left(h^2\right)\right]h}{\Delta}$$

Taking the limit and using $(1-p)\frac{h^2}{\Delta}\to\frac{\sigma^2}{2}$ and $Q'\left(\underline{y}\right)=0$ implies that the outflow is $\frac{\sigma^2}{2}Q''\left(\underline{y}\right)$. Finally, $Q(y)=\tilde{G}_a\left(e^y\right)$ implies $Q'(y)=e^y\tilde{G}'_a\left(e^y\right)$ and further $Q''(y)=e^y\tilde{G}'_a\left(e^y\right)+\left(e^y\right)^2\tilde{G}''_a\left(e^y\right)$. Using $\tilde{G}'_a\left(\underline{x}_a\right)=0$, we that the outflow is $\frac{\sigma^2}{2}\underline{x}_a^2\tilde{G}''_a\left(\underline{x}_a\right)$. Thus the change in the measure of plants is

$$\tilde{N}(\infty,t)-\frac{\sigma^2}{2}\underline{x}_{a,t}^2\tilde{G}''_a\left(\underline{x}_a\right)=\frac{d}{dt}\left[\tilde{G}_{a,t}(\infty)\right]$$

ot, dividing by J_t, using $\tilde{G}_{a,t}=J_tG_{a,t}$ which implies $\frac{d}{dt}\tilde{G}_{a,t}=\dot{J}_tG_{a,t}+J_t\dot{G}_{a,t}$

$$N(\infty,t)-\frac{\sigma^2}{2}\underline{x}_a^2G''_a\left(\underline{x}_a\right)=\frac{1}{J_t}\frac{d}{dt}\left[\tilde{G}_{a,t}(\infty)\right]=\frac{1}{J_t}\frac{d}{dt}\left[\tilde{G}_{a,t}(\infty)\right]=\frac{\dot{J}_tG_{a,t}(\infty)+J\dot{G}_{a,t}(\infty)}{J}$$

Finally, along a balanced growth path, the normalied distributions are constant and $\dot{J}_t/J_t=g_L$, giving

$$N(\infty)-\frac{\sigma^2}{2}\underline{x}_a^2G''_a\left(\underline{x}_a\right)=g_LG_a(\infty)$$

■

D.2 Solving the KFE

Recall that the KFE is

$$0=-g_LG'_a(x)-\frac{d}{dx}\left[\mu xG'_a(x)\right]+\frac{d^2}{dx^2}\left[\frac{\sigma^2}{2}x^2G'_a(x)\right]+N'_a(x)-B_ax^{-\zeta}G'_a(x)+B_a\zeta x^{-\zeta-1}G_a(x)$$

with initial and terminal conditions

$$\begin{aligned}G_a\left(\underline{x}_a\right)&=0\\G'_a\left(\underline{x}_a\right)&=0\\\frac{\sigma^2}{2}\underline{x}_a^2G''_a\left(\underline{x}_a\right)&=N_a(\infty)-g_LG_a(\infty)\end{aligned}$$

and $N_a(x)\equiv\eta_a\left[1-\left(\frac{x}{\underline{x}_a}\right)^{-\zeta}\right]$ and $\eta_a\equiv\Upsilon F'_a(a)\,\underline{x}_a^{-\zeta}\int h(q,a)^{\zeta}\,n(q)\,dF_q(q)$

Integrate from $\underline{x}_a$ to x and using $G_a(\underline{x}_a) = G'_a(\underline{x}_a) = 0$

$$0 = -g_L G_a(x) - \mu x G'_a(x) + \frac{\sigma^2}{2} 2x G'_a(x) + \frac{\sigma^2}{2} x^2 G''_a(x) - \frac{\sigma^2}{2} \underline{x}_a^2 G''_a(\underline{x}_a) + N_a(x) + B_a \left[-\int_{\underline{x}_a}^{x} G'_a(u) u^{-\zeta} du + \int_{\underline{x}_a}^{x} \zeta u^{-\zeta-1} G_a(\right.$$

Integrating by parts, we have

$$\begin{aligned} \int_{\underline{x}_a}^{x} \zeta u^{-\zeta-1} G_a(u) du &= -u^{-\zeta} G_a(u)\Big|_{\underline{x}_a}^{x} + \int_{\underline{x}_a}^{x} u^{-\zeta} G'_a(u) du \\ &= -x^{-\zeta} G_a(x) + \int_{\underline{x}_a}^{x} u^{-\zeta} G'_a(u) du \end{aligned}$$

This, along with the boundary condition $N_a(\infty) = \frac{\sigma^2}{2} \underline{x}_a^2 G''_a(\underline{x}_a) + g_L G_a(\infty)$ gives

$$\begin{aligned} N_a(\infty) - N_a(x) &= g_L [G_a(\infty) - G_a(x)] - [\mu - \sigma^2] x G'_a(x) + \frac{\sigma^2}{2} x^2 G''_a(x) - B_a x^{-\zeta} G_a(x) \\ N_a(\infty) - N_a(x) + B_a x^{-\zeta} G_a(\infty) &= g_L [G_a(\infty) - G_a(x)] - [\mu - \sigma^2] x G'_a(x) + \frac{\sigma^2}{2} x^2 G''_a(x) + B_a x^{-\zeta} [G_a(\infty) - G_a(\end{aligned}$$

Using $N_a(x) \equiv \eta_a \left[1 - \left(\frac{x}{\underline{x}_a}\right)^{-\zeta}\right]$ and defining $b_a = B_a \underline{x}_a^{-\zeta}$, this is

$$\eta_a \left(\frac{x}{\underline{x}_a}\right)^{-\zeta} + B_a x^{-\zeta} G_a(\infty) = g_L [G_a(\infty) - G_a(x)] - [\mu - \sigma^2] x G'_a(x) + \frac{\sigma^2}{2} x^2 G''_a(x) + B_a x^{-\zeta} [G_a(\infty) - G_a(x)]$$

Define the variables

$$\begin{aligned} \phi_a &\equiv \left(\frac{b_a}{\frac{\sigma^2}{2} \left(\frac{\zeta}{2}\right)^2} \right)^{1/2} \\ \tilde{k} &\equiv \frac{1}{\zeta} \left(1 - \frac{\mu}{\sigma^2/2}\right) \end{aligned}$$

and define $\tilde{\varpi}$ to be the positive root of

$$\tilde{\varpi}^2 = \tilde{k}^2 + \frac{g_L}{\frac{\sigma^2}{2} \left(\frac{\zeta}{2}\right)^2}$$

Note that

$$\begin{aligned}\tilde{\varpi}+\tilde{k} &= \sqrt{\left(\frac{1}{\zeta}\left(1-\frac{\mu}{\sigma^2/2}\right)\right)^2+\frac{g_L}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}}+\frac{1}{\zeta}\left(1-\frac{\mu}{\sigma^2/2}\right) \\ &= \frac{1}{\zeta}\left[\sqrt{\left(1-\frac{\mu}{\sigma^2/2}\right)^2+\frac{g_L}{\frac{\sigma^2}{2}\left(\frac{1}{2}\right)^2}}+\left(1-\frac{\mu}{\sigma^2/2}\right)\right]\end{aligned}$$

and the function

$$\Gamma_a(t) \equiv t^{-\tilde{k}}\frac{G_a(\infty)-G_a\left(\underline{x}_a\phi_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)}{\frac{\eta_a}{b_a}+G_a(\infty)}$$

One can verify that $\Gamma_a(t)$ satisfies the differential equation

$$-t^{2-\tilde{k}} = -\left(\tilde{\varpi}^2+t^2\right)\Gamma_a(t)+t\Gamma_a'(t)+t^2\Gamma_a''(t) \tag{41}$$

with boundary conditions

$$\Gamma_a(\phi_a) \equiv \frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\phi_a^{-\tilde{k}} \tag{42}$$

$$\Gamma_a'(\phi_a) = -\frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\tilde{k}\phi_a^{-\tilde{k}-1} \tag{43}$$

$$\Gamma_a''(\phi_a) = \frac{\left(\tilde{k}+1\right)\tilde{k}G_a(\infty)-\frac{\eta_a-g_LG_a(\infty)}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}}{\frac{\eta_a}{b_a}+G_a(\infty)}\phi_a^{-\tilde{k}-2} \tag{44}$$

Define the functions

$$\begin{aligned}\tilde{\theta}(\phi) &= \frac{1}{\phi I_{\tilde{\varpi}}(\phi)^2}\int_0^{\phi}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du \\ \tilde{\Theta}(x_1,x_2) &= \int_{x_2}^{x_1}\tilde{\theta}(u)\,du\end{aligned}$$

Proposition D.2 *Defined as*

$$\Gamma(t,\phi) \equiv I_{\tilde{\varpi}}(t)\left\{\tilde{\Theta}(\phi,t)+\frac{\phi I_{\tilde{\varpi}}(\phi)\,\tilde{\theta}(\phi)}{\tilde{k}I_{\tilde{\varpi}}(\phi)+\phi I_{\tilde{\varpi}}'(\phi)}\right\}$$

The solution to each (41) subject to (42), (43), and (44) is

$$\Gamma_a(t) = \Gamma(t, \phi_a)$$

In addition,

$$\frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} = \phi_a^{\tilde{k}} \frac{\int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du}{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)}$$

Proof. The two fundamental solutions to are the $I_{\tilde{\varpi}}(t)$ and $K_{\tilde{\varpi}}(t)$ and a variation of parameters yields the particular solution

$$\Gamma_{ap}(t) = I_{\tilde{\varpi}}(t) \int_t^{\phi_a} u^{1-\tilde{k}} K_{\tilde{\varpi}}(u)\, du + K_{\tilde{\varpi}}(t) \int_{\phi_a}^{t} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du$$

The general solution is thus

$$\Gamma_a(t) = C_1 I_{\varpi}(t) + C_2 K_{\varpi}(t) + I_{\varpi}(t) \int_t^{\phi_a} u^{1-k} K_{\varpi}(u)\, du + K_{\varpi}(t) \int_{\phi_a}^{t} u^{1-k} I_{\varpi}(u)\, du$$

We next impose the boundary conditions. Note that the third (44) holds by construction if (42) and (43) hold. Differentiating gives

$$\begin{aligned}
\Gamma_a(t) &= C_1 I_{\tilde{\varpi}}(t) + C_2 K_{\tilde{\varpi}}(t) + I_{\tilde{\varpi}}(t) \int_t^{\phi_a} u^{1-\tilde{k}} K_{\tilde{\varpi}}(u)\, du + K_{\tilde{\varpi}}(t) \int_{\phi_a}^{t} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du \\
\Gamma'_a(t) &= C_1 I'_{\tilde{\varpi}}(t) + C_2 K'_{\tilde{\varpi}}(t) + I'_{\tilde{\varpi}}(t) \int_t^{\phi_a} u^{1-\tilde{k}} K_{\tilde{\varpi}}(u)\, du + K'_{\tilde{\varpi}}(t) \int_{\phi_a}^{t} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du
\end{aligned}$$

Evaluating these at $t = \phi_a$ gives

$$\begin{aligned}
\Gamma_a(\phi_a) &= C_1 I_{\tilde{\varpi}}(\phi_a) + C_2 K_{\tilde{\varpi}}(\phi_a) \\
\Gamma'_a(\phi_a) &= C_1 I'_{\tilde{\varpi}}(\phi_a) + C_2 K'_{\tilde{\varpi}}(\phi_a)
\end{aligned}$$

(42) and (43) can thus be expressed as

$$\begin{aligned}
\phi_a^{-\tilde{k}} \frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} &= C_1 I_{\tilde{\varpi}}(\phi_a) + C_2 K_{\tilde{\varpi}}(\phi_a) \\
-\tilde{k} \phi_a^{-\tilde{k}-1} \frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} &= C_1 I'_{\tilde{\varpi}}(\phi_a) + C_2 K'_{\tilde{\varpi}}(\phi_a)
\end{aligned}$$

Solving for the two constants gives

$$\begin{aligned} C_1 &= -\frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\phi_a^{-\tilde{k}}\left[\tilde{k}K_{\tilde{\varpi}}(\phi_a)+\phi_a K'_{\tilde{\varpi}}(\phi_a)\right] \\ C_2 &= \phi_a^{-\tilde{k}}\frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\left[\tilde{k}I_{\tilde{\varpi}}(\phi_a)+\phi_a I'_{\tilde{\varpi}}(\phi_a)\right] \end{aligned}$$

Together, these imply that

$$\begin{aligned} \Gamma_a(t) &= \frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\phi_a^{-\tilde{k}}\left\{-\left[\tilde{k}K_{\tilde{\varpi}}(\phi_a)+\phi_a K'_{\tilde{\varpi}}(\phi_a)\right]I_{\tilde{\varpi}}(t)+\left[\tilde{k}I_{\tilde{\varpi}}(\phi_a)+\phi_a I'_{\tilde{\varpi}}(\phi_a)\right]K_{\tilde{\varpi}}(t)\right\} \\ &\quad +I_{\tilde{\varpi}}(t)\int_t^{\phi_a}u^{1-\tilde{k}}K_{\tilde{\varpi}}(u)\,du+K_{\tilde{\varpi}}(t)\int_{\phi_a}^{t}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du \end{aligned}$$

We can rearrange this as

$$\begin{aligned} \Gamma_a(t) &= -\frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\phi_a^{-\tilde{k}}\left[\tilde{k}K_{\tilde{\varpi}}(\phi_a)+\phi_a K'_{\tilde{\varpi}}(\phi_a)\right]I_{\tilde{\varpi}}(t)+I_{\tilde{\varpi}}(t)\int_t^{\phi_a}u^{1-\tilde{k}}K_{\tilde{\varpi}}(u)\,du+K_{\tilde{\varpi}}(t)\int_0^{t}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du \\ &\quad +\left\{\frac{G_a(\infty)}{\frac{\eta_a}{b_a}+G_a(\infty)}\phi_a^{-\tilde{k}}\left[\tilde{k}I_{\tilde{\varpi}}(\phi_a)+\phi_a I'_{\tilde{\varpi}}(\phi_a)\right]-\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du\right\}K_{\tilde{\varpi}}(t) \end{aligned}$$

We must have $\lim_{t\to 0} t^{\tilde{k}}\Gamma_a(t) = \frac{G_a(\infty)-G_a\left(\underline{x}_a\phi_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)}{\frac{\eta_a}{b_a}+G_a(\infty)} = 0$. This will pin down $G_a(\infty)$.

We first show that the limits of the products of $t^{\tilde{k}}$ and each of the terms of the top line are zero. Note that $I_{\tilde{\varpi}}(t)\sim\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}}$ and $K_{\tilde{\varpi}}(t)\sim\frac{1}{2}\Gamma(\tilde{\varpi})\left(\frac{t}{2}\right)^{-\tilde{\varpi}}$ as $t\to 0$. Then we have

$$\begin{aligned} \lim_{t\to 0} t^{\tilde{k}}I_{\tilde{\varpi}}(t) &= \lim_{t\to 0} t^{\tilde{k}}\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}}\frac{I_{\tilde{\varpi}}(t)}{\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}}} \\ &= \lim_{t\to 0} t^{\tilde{k}}\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}} = 0 \end{aligned}$$

We next turn to the term $t^{\tilde{k}}I_{\tilde{\varpi}}(t)\int_t^{\phi_a}u^{1-\tilde{k}}K_{\tilde{\varpi}}(u)\,du$. If $2\geq\tilde{\varpi}+\tilde{k}$, then $\lim_{t\to\infty}\int_t^{\phi_a}u^{1-\tilde{k}}K_{\tilde{\varpi}}(u)\,du$ is finite and $\lim_{t\to 0}t^{\tilde{k}}I_{\tilde{\varpi}}(t)=0$. If $2<\tilde{\varpi}+\tilde{k}$ then

$$\begin{aligned} \lim_{t\to 0} t^{\tilde{k}}I_{\tilde{\varpi}}(t)\int_t^{\phi_a}u^{1-\tilde{k}}K_{\tilde{\varpi}}(u)\,du &= \lim_{t\to 0} t^{\tilde{k}}\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}}\left[-\frac{\frac{1}{2}\Gamma(\tilde{\varpi})\left(\frac{t}{2}\right)^{-\tilde{\varpi}}t^{2-\tilde{k}}}{2-\tilde{\varpi}-\tilde{k}}\right]\frac{I_{\tilde{\varpi}}(t)}{\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}}}\frac{\int_t^{\phi_a}u^{1-\tilde{k}}K_{\tilde{\varpi}}(u)\,du}{-\frac{\frac{1}{2}\Gamma(\tilde{\varpi})\left(\frac{t}{2}\right)^{-\tilde{\varpi}}t^{2-\tilde{k}}}{2-\tilde{\varpi}-\tilde{k}}} \\ &= \lim_{t\to 0}\frac{\Gamma(\tilde{\varpi})}{\Gamma(1+\tilde{\varpi})}\frac{\frac{1}{2}t^2}{\tilde{\varpi}+\tilde{k}-2} = 0 \end{aligned}$$

$$\lim_{t\to 0} t^{\tilde{k}} K_{\tilde{\varpi}}(t) \int_0^t u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du = \lim_{t\to 0} t^{\tilde{k}} \frac{1}{2}\Gamma(\tilde{\varpi}) \left(\frac{t}{2}\right)^{-\tilde{\varpi}} \frac{\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}} t^{2-\tilde{k}}}{\tilde{\varpi}+2-\tilde{k}} \frac{K_{\tilde{\varpi}}(t)}{\frac{1}{2}\Gamma(\tilde{\varpi})\left(\frac{t}{2}\right)^{-\tilde{\varpi}}} \frac{\int_0^t u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du}{\frac{\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}} t^{2-\tilde{k}}}{\tilde{\varpi}+2-\tilde{k}}}$$

$$= \lim_{t\to 0} t^{\tilde{k}} \frac{1}{2}\Gamma(\tilde{\varpi}) \left(\frac{t}{2}\right)^{-\tilde{\varpi}} \frac{\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}} t^{2-\tilde{k}}}{\tilde{\varpi}+2-\tilde{k}} = \lim_{t\to 0} \frac{1}{2} \frac{\frac{\Gamma(\tilde{\varpi})}{\Gamma(1+\tilde{\varpi})} t^2}{\tilde{\varpi}+2-\tilde{k}} = 0$$

Next, we have

$$\lim_{t\to 0} t^{\tilde{k}} K_{\tilde{\varpi}}(t) = \lim_{t\to 0} t^{\tilde{k}} \frac{1}{2}\Gamma(\tilde{\varpi}) \left(\frac{t}{2}\right)^{-\tilde{\varpi}} \frac{K_{\tilde{\varpi}}(t)}{\frac{1}{2}\Gamma(\tilde{\varpi})\left(\frac{t}{2}\right)^{-\tilde{\varpi}}} = \lim_{t\to 0} t^{\tilde{k}} \frac{1}{2}\Gamma(\tilde{\varpi}) \left(\frac{t}{2}\right)^{-\tilde{\varpi}} = \infty$$

Since $\tilde{\varpi} \geq \tilde{k}$, this limit is strictly positive (it is inifinite if $g_L > 0$ which implies $\tilde{\varpi} > \tilde{k}$, or finite if $g_L = 0$ which implies $\tilde{\varpi} = \tilde{k}$). So the only way $\lim_{t\to 0} t^{\tilde{k}} \Gamma_a(t) = 0$ is if

$$0 = \frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} \phi_a^{-\tilde{k}} \left[\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)\right] - \int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du$$

This implies that

$$\frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} = \phi_a^{\tilde{k}} \frac{\int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du}{\left[\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)\right]}$$

and that we can express Γ_a as

$$\Gamma_a(t) = \Gamma(t, \phi_a) \equiv -I_{\tilde{\varpi}}(t) \left\{ \frac{\tilde{k} K_{\tilde{\varpi}}(\phi_a) + \phi_a K'_{\tilde{\varpi}}(\phi_a)}{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)} \int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du - \int_t^{\phi_a} u^{1-\tilde{k}} K_{\tilde{\varpi}}(u)\, du \right\} + K_{\tilde{\varpi}}(t) \int_0^t u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)$$

Using the Wronskian of the modified Bessel functions, we have ■

$$\frac{\tilde{k} K_{\tilde{\varpi}}(\phi) + \phi K'_{\tilde{\varpi}}(\phi)}{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)} I_{\tilde{\varpi}}(\phi) - K_{\tilde{\varpi}}(\phi) = \frac{\tilde{k} I_{\tilde{\varpi}}(\phi) K_{\tilde{\varpi}}(\phi) + \phi I_{\tilde{\varpi}}(\phi) K'_{\tilde{\varpi}}(\phi) - \tilde{k} I_{\tilde{\varpi}}(\phi) K_{\tilde{\varpi}}(\phi) - \phi I'_{\tilde{\varpi}}(\phi) K_{\tilde{\varpi}}(\phi)}{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)}$$

$$= \frac{\phi \left[I_{\tilde{\varpi}}(\phi) K'_{\tilde{\varpi}}(\phi) - I'_{\tilde{\varpi}}(\phi) K_{\tilde{\varpi}}(\phi)\right]}{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)}$$

$$= -\frac{1}{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)}$$

or

$$\frac{\tilde{k} K_{\tilde{\varpi}}(\phi) + \phi K'_{\tilde{\varpi}}(\phi)}{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)} = -\frac{1}{I_{\tilde{\varpi}}(\phi)} \frac{1}{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)} + \frac{K_{\tilde{\varpi}}(\phi)}{I_{\tilde{\varpi}}(\phi)}$$

Plugging this in gives

$$\begin{aligned}\Gamma\left(t,\phi\right) &= -I_{\tilde{\varpi}}\left(t\right)\left\{\left\{-\frac{1}{I_{\tilde{\varpi}}\left(\phi\right)}\frac{1}{\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)}+\frac{K_{\tilde{\varpi}}\left(\phi\right)}{I_{\tilde{\varpi}}\left(\phi\right)}\right\}\int_0^{\phi}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du-\int_t^{\phi}u^{1-\tilde{k}}K_{\tilde{\varpi}}\left(u\right)du\right\}+K_{\tilde{\varpi}}\left(t\right)\int_0^t u \\ &= I_{\tilde{\varpi}}\left(t\right)\left\{\frac{1}{I_{\tilde{\varpi}}\left(\phi\right)}\frac{\int_0^{\phi}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du}{\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)}-\frac{K_{\tilde{\varpi}}\left(\phi\right)}{I_{\tilde{\varpi}}\left(\phi\right)}\int_0^{\phi}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du+\int_t^{\phi}u^{1-\tilde{k}}K_{\tilde{\varpi}}\left(u\right)du+\frac{K_{\tilde{\varpi}}\left(t\right)}{I_{\tilde{\varpi}}\left(t\right)}\int_0^t u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right.\right.\end{aligned}$$

Using the fundamental theorem of calculus, the Wronskian identity, and the definition of $\tilde{\theta}$ gives

$$\begin{aligned}\Gamma_a\left(t\right) &= I_{\tilde{\varpi}}\left(t\right)\left\{\frac{\phi I_{\tilde{\varpi}}\left(\phi\right)\tilde{\theta}\left(\phi\right)}{\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)}+\int_t^{\phi}\frac{d}{dw}\left\{-\frac{K_{\tilde{\varpi}}\left(w\right)}{I_{\tilde{\varpi}}\left(w\right)}\int_0^{w}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du+\int_{\infty}^{w}u^{1-\tilde{k}}K_{\tilde{\varpi}}\left(u\right)du\right\}dw\right\} \\ &= I_{\tilde{\varpi}}\left(t\right)\left\{\frac{\phi I_{\tilde{\varpi}}\left(\phi\right)\tilde{\theta}\left(\phi\right)}{\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)}+\int_t^{\phi}\left[\frac{-I_{\tilde{\varpi}}\left(w\right)K'_{\tilde{\varpi}}\left(w\right)+I_{\tilde{\varpi}}\left(w\right)K_{\tilde{\varpi}}\left(w\right)}{I_{\tilde{\varpi}}\left(w\right)^2}\right]\int_0^{w}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)dudw\right\} \\ &= I_{\tilde{\varpi}}\left(t\right)\left\{\frac{\phi I_{\tilde{\varpi}}\left(\phi\right)\tilde{\theta}\left(\phi\right)}{\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)}+\int_t^{\phi}\frac{1}{wI_{\tilde{\varpi}}\left(w\right)^2}\int_0^{w}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)dudw\right\} \\ &= I_{\tilde{\varpi}}\left(t\right)\left\{\frac{\phi I_{\tilde{\varpi}}\left(\phi\right)\tilde{\theta}\left(\phi\right)}{\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)}+\tilde{\Theta}\left(\phi,t\right)\right\}\end{aligned}$$

Claim D.3 *The mass of plants in equilibrium that are type a is*

$$G_a\left(\infty\right)=\frac{\eta_a}{g_L}\frac{\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du}{\phi_a^2\int_0^{\phi_a}u^{-\tilde{k}-1}I_{\tilde{\varpi}}\left(u\right)du}$$

Proof. Noting that $\frac{\frac{b_a}{\eta_a}G_a(\infty)}{1+\frac{b_a}{\eta_a}G_a(\infty)}=\frac{\phi_a^{\tilde{k}}\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)du}{\tilde{k}I_{\tilde{\varpi}}(\phi_a)+\phi_a I'_{\tilde{\varpi}}(\phi_a)}$, we have

$$\frac{b_a}{\eta_a}G_a\left(\infty\right)=\frac{\phi_a^{\tilde{k}}\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du}{\tilde{k}I_{\tilde{\varpi}}\left(\phi_a\right)+\phi_a I'_{\tilde{\varpi}}\left(\phi_a\right)-\phi_a^{\tilde{k}}\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du}$$

Define $H_1\left(\phi\right)\equiv\phi^{-\tilde{k}}\left[\tilde{k}I_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)-\phi^{\tilde{k}}\int_0^{\phi}u^{1-\tilde{k}}I_{\tilde{\varpi}}\left(u\right)du\right]$. Recall that by definition, $I_{\tilde{\varpi}}$ satisfies

$$\phi^2 I''_{\tilde{\varpi}}\left(\phi\right)+\phi I'_{\tilde{\varpi}}\left(\phi\right)-\left(\phi^2+\tilde{\varpi}^2\right)I_{\tilde{\varpi}}\left(\phi\right)=0$$

Then we have

$$
\begin{aligned}
\frac{d}{d\phi}\left\{\tilde{k}\phi^{-\tilde{k}}I_{\tilde{\varpi}}(\phi)+\phi^{1-\tilde{k}}I'_{\tilde{\varpi}}(\phi)\right\} &= -\tilde{k}^2\phi^{-\tilde{k}-1}I_{\tilde{\varpi}}(\phi)+\tilde{k}\phi^{-\tilde{k}}I'_{\tilde{\varpi}}(\phi)+\left(1-\tilde{k}\right)\phi^{-\tilde{k}}I'_{\tilde{\varpi}}(\phi)+\phi^{1-\tilde{k}}I''_{\tilde{\varpi}}(\phi)\\
&= \phi^{-\tilde{k}-1}\left[-\tilde{k}^2I_{\tilde{\varpi}}(\phi)+\phi I'_{\tilde{\varpi}}(\phi)+\phi^2I''_{\tilde{\varpi}}(\phi)\right]\\
&= \phi^{-\tilde{k}-1}\left[-\tilde{k}^2I_{\tilde{\varpi}}(\phi)+\left(\phi^2+\tilde{\varpi}^2\right)I_{\tilde{\varpi}}(\phi)\right]\\
&= \phi^{-\tilde{k}-1}\left[\tilde{\varpi}^2-\tilde{k}^2+\phi^2\right]I_{\tilde{\varpi}}(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H_1'(\phi) &\equiv \frac{d}{d\phi}\left\{\tilde{k}\phi^{-\tilde{k}}I_{\tilde{\varpi}}(\phi)+\phi^{1-\tilde{k}}I'_{\tilde{\varpi}}(\phi)-\int_0^\phi u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du\right\}\\
&= \phi^{-\tilde{k}-1}\left[\tilde{\varpi}^2-\tilde{k}^2+\phi^2\right]I_{\tilde{\varpi}}(\phi)-\phi^{1-\tilde{k}}I_{\tilde{\varpi}}(\phi)\\
&= \phi^{-\tilde{k}-1}\left(\tilde{\varpi}^2-\tilde{k}^2\right)I_{\tilde{\varpi}}(\phi)
\end{aligned}
$$

If $g_L>0$ so that $\tilde{\varpi}>\tilde{k}$, then $H_1(0)=0$, and thus

$$
\begin{aligned}
H_1(\phi) &= H_1(0)+\int_0^\phi H_1'(u)\,du\\
&= \left(\tilde{\varpi}^2-\tilde{k}^2\right)\int_0^\phi u^{-\tilde{k}-1}I_{\tilde{\varpi}}(u)\,du
\end{aligned}
$$

and

$$
\frac{b_a}{\eta_a}G_a(\infty)=\frac{\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}{H_1(\phi_a)}=\frac{1}{\tilde{\varpi}^2-\tilde{k}^2}\frac{\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}{\int_0^{\phi_a}u^{-\tilde{k}-1}I_{\tilde{\varpi}}(u)\,du}
$$

Using $\phi_a^2\equiv\frac{b_a}{\frac{\sigma^2}{2}\left(\frac{\varsigma}{2}\right)^2}$ and $\tilde{\varpi}^2=\tilde{k}^2+\frac{g_L}{\frac{\sigma^2}{2}\left(\frac{\varsigma}{2}\right)^2}$, this is

$$
\frac{1}{\eta_a}G_a(\infty)=\frac{1}{g_L}\frac{\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}{\phi_a^2\int_0^{\phi_a}u^{-\tilde{k}-1}I_{\tilde{\varpi}}(u)\,du}
$$

If $g_L=0$ and $\tilde{\varpi}=\tilde{k}$ then

$$
I_{\tilde{\varpi}}(t)\sim\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}t\right)^{\tilde{\varpi}}
$$

$$
\begin{aligned}
\lim_{\phi\to0}\phi^{-\tilde{k}}I_{\tilde{\varpi}}(\phi) &= \lim_{\phi\to0}\frac{I_{\tilde{\varpi}}(\phi)}{\phi^{\tilde{\varpi}}}=\frac{1}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}\right)^{\tilde{\varpi}}\\
\lim_{\phi\to0}\phi^{-\tilde{k}}\phi I'_{\tilde{\varpi}}(\phi) &= \lim_{\phi\to0}\frac{I'_{\tilde{\varpi}}(\phi)}{\phi^{\tilde{\varpi}-1}}=\frac{\tilde{\varpi}}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}\right)^{\tilde{\varpi}}
\end{aligned}
$$

$$\begin{aligned}
\lim_{\phi\to 0} H_1(\phi) &\equiv \phi^{-\tilde{k}}\left[\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi) - \phi^{\tilde{k}}\int_0^{\phi} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\,du\right] \\
&= \frac{\tilde{k}}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}\right)^{\tilde{\varpi}} + \frac{\tilde{\varpi}}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}\right)^{\tilde{\varpi}} \\
&= \frac{2\tilde{\varpi}}{\Gamma(1+\tilde{\varpi})}\left(\frac{1}{2}\right)^{\tilde{\varpi}} \\
&= \frac{2^{1-\tilde{\varpi}}}{\Gamma(\tilde{\varpi})}
\end{aligned}$$

Then

$$\begin{aligned}
H_1(\phi) &= H_1(0) + \int_0^{\phi} H'_1(u)\,du \\
&= \frac{2^{1-\tilde{\varpi}}}{\Gamma(\tilde{\varpi})} + \left(\tilde{\varpi}^2 - \tilde{k}^2\right)\int_0^{\phi} u^{-\tilde{k}-1} I_{\tilde{\varpi}}(u)\,du \\
&= \frac{2^{1-\tilde{\varpi}}}{\Gamma(\tilde{\varpi})}
\end{aligned}$$

$$\frac{b_a}{\eta_a} G_a(\infty) = \frac{\Gamma(\tilde{\varpi})}{2^{1-\tilde{\varpi}}}\int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\,du$$

Using $\phi_a^2 \equiv \frac{b_a}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}$ and $\tilde{\varpi}^2 = \tilde{k}^2 + \frac{g_L}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2}$, this is

$$\frac{1}{\eta_a} G_a(\infty) = \frac{\Gamma(\tilde{\varpi})}{2^{1-\tilde{\varpi}}} \frac{1}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2 \phi_a^2}\int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\,du$$

■

E The Value of Grafting

Another useful expression would be

$$\begin{aligned}
\int_{\underline{x}_a}^{\infty}\int_u^{\infty}\left[v_a(x) - v_a(u)\right]\zeta x^{-\zeta-1}dx dG_a(u) &= \int_u^{\infty}\left[v_a(x) - v_a(u)\right]\zeta x^{-\zeta-1}dx G_a(u)\Big|_{\underline{x}_a}^{\infty} + \int_{\underline{x}_a}^{\infty}\int_u^{\infty} v'_a(u)\zeta x^{-\zeta-1}dx G_a \\
&= \int_{\underline{x}_a}^{\infty} v'_a(u) u^{-\zeta} G_a(u)\,du
\end{aligned}$$

To get at this we will use

$$y_a(t) \equiv \frac{1}{\bar{\pi}}\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^2 t^{-k} v'_a\left(\underline{x}_a \lambda_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}\right)$$

and, from the KFE file,

$$\frac{G_a(\infty) - G_a\left(\underline{x}_a \phi_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}\right)}{\frac{\eta_a}{b_a} + G_a(\infty)} = t^{\tilde{k}} \Gamma_a(t) = t^{\tilde{k}} I_{\tilde{\varpi}}(t) \left\{ \tilde{\Theta}(\phi_a) - \tilde{\Theta}(t) + \frac{1}{I_{\tilde{\varpi}}(\phi_a)} \frac{\int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du}{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)} \right\}$$

which, along with

$$\frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} = \frac{\phi_a^{\tilde{k}} \int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du}{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)}$$

$$\begin{aligned} \frac{G_a(\infty) - G_a\left(\underline{x}_a \phi_a^{\frac{2}{\zeta}} t^{-\frac{2}{\zeta}}\right)}{G_a(\infty)} &= \frac{\frac{\eta_a}{b_a} + G_a(\infty)}{G_a(\infty)} t^{\tilde{k}} I_{\tilde{\varpi}}(t) \left\{ \tilde{\Theta}(\phi_a, t) + \frac{1}{I_{\tilde{\varpi}}(\phi_a)} \frac{\int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du}{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)} \right\} \\ &= t^{\tilde{k}} I_{\tilde{\varpi}}(t) \left\{ \frac{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)}{\phi_a^{\tilde{k}} \int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du} \tilde{\Theta}(\phi_a, t) + \frac{1}{\phi_a^{\tilde{k}}} \frac{1}{I_{\tilde{\varpi}}(\phi_a)} \right\} \end{aligned}$$

Claim E.1 *We can express the integral as*

$$\int_{\underline{x}_a}^{\infty} v'_a(u)\, u^{-\zeta} G_a(u)\, du = \frac{\bar{\pi} \underline{x}_a^{1-\zeta}}{\frac{\sigma^2}{2} \left(\frac{\zeta}{2}\right)^3} G_a(\infty) \left\{ \lambda_a^{\frac{2}{\zeta}-2} T(\lambda_a) + A_1(\lambda_a) + Q(\phi_a) A_2(\lambda_a) \right\}$$

where $G_a(\infty)$ *satisfies* $\frac{G_a(\infty)}{\frac{\eta_a}{b_a} + G_a(\infty)} = \frac{\phi_a^{\tilde{k}} \int_0^{\phi_a} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u) du}{\tilde{k} I_{\tilde{\varpi}}(\phi_a) + \phi_a I'_{\tilde{\varpi}}(\phi_a)}$ *and* $A_1(\cdot)$, $A_2(\cdot)$, *and* $Q(\cdot)$ *are defined using* $\varrho \equiv \frac{\phi_a}{\lambda_a} = \frac{1}{\sqrt{1-\beta}}$ *as*

$$\begin{aligned} A_1(\lambda) &\equiv \frac{\int_0^{\lambda} t^k I_{\varpi}(t) \Theta(\lambda, t) (\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\, t^{2-\frac{2}{\zeta}-1} dt}{\lambda^{2-\frac{2}{\zeta}} (\rho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)} \\ A_2(\lambda) &\equiv \frac{\int_0^{\lambda} t^k I_{\varpi}(t) \Theta(\lambda, t) (\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t) \left[\tilde{\Theta}(\varrho\lambda) - \tilde{\Theta}(\varrho t)\right] t^{2-\frac{2}{\zeta}-1} dt}{(\varrho\lambda)^2 \lambda^{2-\frac{2}{\zeta}}} \\ Q(\phi) &\equiv \phi^2 \frac{\tilde{k} I_{\tilde{\varpi}}(\phi) + \phi I'_{\tilde{\varpi}}(\phi)}{\phi^{\tilde{k}} \int_0^{\phi} u^{1-\tilde{k}} I_{\tilde{\varpi}}(u)\, du} \end{aligned}$$

Proof. We start by rearranging:

$$\int_{\underline{x}_a}^{\infty} v'_a(u)\, u^{-\zeta} G_a(u)\, du = G_a(\infty) \int_{\underline{x}_a}^{\infty} v'_a(u)\, u^{-\zeta} du - G_a(\infty) \int_{\underline{x}_a}^{\infty} v'_a(u)\, u^{-\zeta} \frac{G_a(\infty) - G_a(u)}{G_a(\infty)} du$$

If we know λ_a and ϕ_a, then we have expressions for $G_a(\infty)$ and $\int_{\underline{x}_a}^{\infty} v'_a(u)\, u^{-\zeta} du$. So we need an expression for the last term. We use the change of variables

$$\begin{aligned}\int_{\underline{x}_a}^{\infty} v_a'(u)\, u^{-\zeta}\frac{G_a(\infty)-G_a(u)}{G_a(\infty)}du &= \int_0^{\lambda_a} v_a'\left(\underline{x}_a\lambda_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)\left(\underline{x}_a\lambda_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)^{-\zeta}\frac{G_a(\infty)-G_a\left(\underline{x}_a\lambda_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)}{G_a(\infty)}\underline{x}_a\lambda_a^{\frac{2}{\zeta}}\frac{2}{\zeta}t^{-\frac{2}{\zeta}-1}dt\\ &= \underline{x}_a^{1-\zeta}\lambda_a^{\frac{2}{\zeta}-2}\frac{2}{\zeta}\int_0^{\lambda_a} v_a'\left(\underline{x}_a\lambda_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)\frac{G_a(\infty)-G_a\left(\underline{x}_a\lambda_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)}{G_a(\infty)}\frac{2}{\zeta}t^{2-\frac{2}{\zeta}-1}dt\end{aligned}$$

Define $\varrho=\frac{\phi_a}{\lambda_a}=\frac{1}{\sqrt{1-\beta}}$. Using $G_a\left(\underline{x}_a\lambda_a^{\frac{2}{\zeta}}t^{-\frac{2}{\zeta}}\right)=G_a\left(\underline{x}_a\phi_a^{\frac{2}{\zeta}}(\varrho t)^{-\frac{2}{\zeta}}\right)$, we can use the expressions for y and Γ. The integral can be expressed as

$$\begin{aligned}\int_{\underline{x}_a}^{\infty} v_a'(u)\, u^{-\zeta}\frac{G_a(\infty)-G_a(u)}{G_a(\infty)}du &= \frac{\underline{x}_a^{1-\zeta}\lambda_a^{\frac{2}{\zeta}-2}\bar{\pi}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3}\left\{\int_0^{\lambda_a} t^k I_{\varpi}(t)\,\Theta(\lambda_a,t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\left\{\frac{\tilde{k}I_{\tilde{\varpi}}(\phi_a)+\phi_a I_{\tilde{\varpi}}'(\phi_a)}{\phi_a^{\tilde{k}}\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}\Big[\tilde{\Theta}(\phi_a)\right.\right.\\ &= \frac{\underline{x}_a^{1-\zeta}\lambda_a^{\frac{2}{\zeta}-2}\bar{\pi}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3}\left\{\int_0^{\lambda_a} t^k I_{\varpi}(t)\,\Theta(\lambda_a,t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\left\{\frac{\tilde{k}I_{\tilde{\varpi}}(\phi_a)+\phi_a I_{\tilde{\varpi}}'(\phi_a)}{\phi_a^{\tilde{k}}\int_0^{\phi_a}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}\Big[\tilde{\Theta}(\phi_a)\right.\right.\\ &= \frac{\underline{x}_a^{1-\zeta}\bar{\pi}}{\frac{\sigma^2}{2}\left(\frac{\zeta}{2}\right)^3}\left[A_1(\lambda_a)+Q(\phi_a)A_2(\lambda)\right]\end{aligned}$$

■

Claim E.2 *A_1 and A_2 satisfy the ODEs*

$$\begin{aligned}\left[2-\frac{2}{\zeta}+\tilde{k}+\frac{\varrho\lambda I_{\tilde{\varpi}}'(\varrho\lambda)}{I_{\tilde{\varpi}}(\varrho\lambda)}\right]A_1(\lambda)+\lambda A_1'(\lambda) &= \frac{\int_0^{\lambda}t^{-k-1}I_{\varpi}(u)\,du}{\lambda^{-k}I_{\varpi}(\lambda)}\frac{\int_0^{\lambda}t^k I_{\varpi}(t)(\varrho t)^{\tilde{k}}I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{\lambda^k I_{\varpi}(\lambda)\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)}\\ \left(4-\frac{2}{\zeta}\right)A_2(\lambda)+\lambda A_2'(\lambda) &= \frac{\int_0^{\lambda}u^{-k-1}I_{\varpi}(u)\,du}{\lambda^{-k}I_{\varpi}(\lambda)}A_3(\lambda)+\frac{\int_0^{\varrho\lambda}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}{(\varrho\lambda)^{2-\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)}A_1(\lambda)\end{aligned}$$

where A_3 is

$$A_3(\lambda)\equiv\frac{\int_0^{\lambda}t^k I_{\varpi}(t)(\varrho t)^{\tilde{k}}I_{\tilde{\varpi}}(\varrho t)\left\{\tilde{\Theta}(\varrho\lambda,\varrho t)\right\}t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^2\lambda^k I_{\varpi}(\lambda)\lambda^{2-\frac{2}{\zeta}}}$$

and satisfies the differential equations

$$\left(4+\frac{\lambda I_{\varpi}'(\lambda)}{I_{\varpi}(\lambda)}+k-\frac{2}{\zeta}\right)A_3(\lambda)+\lambda A_3'(\lambda)=\frac{\int_0^{\varrho\lambda}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}{(\varrho\lambda)^{2-\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)}\frac{\int_0^{\lambda}t^k I_{\varpi}(t)(\varrho t)^{\tilde{k}}I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^{\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)\lambda^k I_{\varpi}(\lambda)\lambda^{2-\frac{2}{\zeta}}}$$

Proof. To solve for A_1 we have

$$\begin{aligned}
A_1(\lambda) &\equiv \frac{\int_0^\lambda t^k I_\varpi(t)\,\Theta(\lambda,t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)} \\
\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)\,A_1(\lambda) &\equiv \int_0^\lambda t^k I_\varpi(t)\,\Theta(\lambda,t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt \\
\frac{d\left[\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)\right]}{d\lambda}A_1(\lambda)+\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)\,A_1'(\lambda) &\equiv \theta(\lambda)\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt \\
\frac{d\log\left[\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)\right]}{d\log\lambda}A_1(\lambda)+\lambda A_1'(\lambda) &\equiv \frac{\lambda\theta(\lambda)\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)} \\
\left[2-\frac{2}{\zeta}+\tilde{k}+\frac{\varrho\lambda I_{\tilde{\varpi}}'(\varrho\lambda)}{I_{\tilde{\varpi}}(\varrho\lambda)}\right]A_1(\lambda)+\lambda A_1'(\lambda) &\equiv \frac{\frac{\int_0^\lambda t^{-k-1}I_\varpi(u)du}{I_\varpi(\lambda)^2}\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)} \\
&= \frac{\int_0^\lambda t^{-k-1}I_\varpi(u)\,du}{\lambda^{-k}I_\varpi(\lambda)}\frac{\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{\lambda^k I_\varpi(\lambda)\,\lambda^{2-\frac{2}{\zeta}}(\varrho\lambda)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho\lambda)}
\end{aligned}$$

To solve for A_2, w have

$$\begin{aligned}
(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}A_2(\lambda) &= \int_0^\lambda t^k I_\varpi(t)\,\Theta(\lambda,t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,\tilde{\Theta}(\varrho\lambda,\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt \\
\frac{d\left[(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}\right]}{d\lambda}A_2(\lambda)+(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}A_2'(\lambda) &= \theta(\lambda)\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,\tilde{\Theta}(\varrho\lambda,\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt+\varrho\tilde{\theta}(\varrho\lambda)\int_0^\lambda t^k I_\varpi(t)\,\Theta \\
\frac{d\ln\left[(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}\right]}{d\ln\lambda}A_2(\lambda)+\lambda A_2'(\lambda) &= \frac{\lambda\theta(\lambda)\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,\tilde{\Theta}(\varrho\lambda,\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}}+\frac{\varrho\lambda\tilde{\theta}(\varrho\lambda)\int_0^\lambda t^k I_\varpi(t)}{(} \\
\left(4-\frac{2}{\zeta}\right)A_2(\lambda)+\lambda A_2'(\lambda) &= \frac{\int_0^\lambda u^{-k-1}I_\varpi(u)\,du}{I_\varpi(\lambda)^2}\frac{\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,\tilde{\Theta}(\varrho\lambda,\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}}+\frac{\int_0^{\varrho\lambda}u^{1-}}{I_{\tilde{\varpi}}} \\
\left(4-\frac{2}{\zeta}\right)A_2(\lambda)+\lambda A_2'(\lambda) &= \frac{\int_0^\lambda u^{-k-1}I_\varpi(u)\,du}{\lambda^{-k}I_\varpi(\lambda)}\frac{\int_0^\lambda t^k I_\varpi(t)\,(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\,\tilde{\Theta}(\varrho\lambda,\varrho t)\,t^{2-\frac{2}{\zeta}-1}dt}{\lambda^k I_\varpi(\lambda)\,(\varrho\lambda)^2\lambda^{2-\frac{2}{\zeta}}}+\frac{\int_0^{\varrho\lambda}u^{1-}}{(\varrho\lambda)^{2-}} \\
&= \frac{\int_0^\lambda u^{-k-1}I_\varpi(u)\,du}{\lambda^{-k}I_\varpi(\lambda)}A_3(\lambda)+\frac{\int_0^{\varrho\lambda}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)\,du}{(\varrho\lambda)^{2-\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)}A_4(\lambda)
\end{aligned}$$

To solve for A_3, we have

$$\begin{aligned}
A_3(\lambda) &\equiv \frac{\int_0^\lambda t^k I_\varpi(t)(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\tilde{\Theta}(\varrho\lambda,\varrho t)t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}} \\
(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}A_3(\lambda) &\equiv \int_0^\lambda t^k I_\varpi(t)(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)\tilde{\Theta}(\varrho\lambda,\rho t)t^{2-\frac{2}{\zeta}-1}dt \\
\frac{d(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}}{d\lambda}A_3(\lambda)+(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}A_3'(\lambda) &\equiv \varrho\tilde{\theta}(\varrho\lambda)\int_0^\lambda t^k I_\varpi(t)(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)t^{2-\frac{2}{\zeta}-1}dt \\
\frac{d\ln(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}}{d\ln\lambda}A_3(\lambda)+\lambda A_3'(\lambda) &\equiv \frac{\varrho\lambda\tilde{\theta}(\varrho\lambda)\int_0^\lambda t^k I_\varpi(t)(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}} \\
\left(4+\frac{\lambda I_\varpi'(\lambda)}{I_\varpi(\lambda)}+k-\frac{2}{\zeta}\right)A_3(\lambda)+\lambda A_3'(\lambda) &= \frac{\int_0^{\varrho\lambda}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)du}{I_{\tilde{\varpi}}(\varrho\lambda)^2}\frac{\int_0^\lambda t^k I_\varpi(t)(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^2\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}} \\
&= \frac{\int_0^{\varrho\lambda}u^{1-\tilde{k}}I_{\tilde{\varpi}}(u)du}{(\varrho\lambda)^{2-\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)}\frac{\int_0^\lambda t^k I_\varpi(t)(\varrho t)^{\tilde{k}} I_{\tilde{\varpi}}(\varrho t)t^{2-\frac{2}{\zeta}-1}dt}{(\varrho\lambda)^{\tilde{k}}I_{\tilde{\varpi}}(\varrho\lambda)\lambda^k I_\varpi(\lambda)\lambda^{2-\frac{2}{\zeta}}}
\end{aligned}$$

■

E.1 Characterization

Lemma E.3 *Suppose that $h(q,a)$ is log-supermodular and $\frac{f_1(a)}{f_2(a)}$ is decreasing in a. Then the ratio $\frac{\int h(q,a)^\zeta f_1(a)da}{\int h(q,a)^\zeta f_2(a)da}$ is decreasing in q.*

Proof. Take logs and differentiate with respect to q.

$$\frac{d\log}{dq}\left\{\frac{\int h(q,a)^\zeta f_1(a)\,da}{\int h(q,a)^\zeta f_2(a)\,da}\right\}=\frac{\int\zeta\frac{d\log h(q,a)}{dq}h(q,a)^\zeta f_1(a)\,da}{\int h(q,a)^\zeta f_1(a)\,da}-\frac{\int\zeta\frac{d\log h(q,a)}{dq}h(q,a)^\zeta f_2(a)\,da}{\int h(q,a)^\zeta f_2(a)\,da}$$

Define $g_i(a,q)=\frac{h(q,a)^\zeta f_i(a)}{\int h(q,\tilde{a})^\zeta f_i(\tilde{a})d\tilde{a}}$. Then this is

$$\frac{d\log}{dq}\left\{\frac{\int h(q,a)^\zeta f_1(a)\,da}{\int h(q,a)^\zeta f_2(a)\,da}\right\}=\int\zeta\frac{d\log h(q,a)}{dq}\left[g_1(a,q)-g_2(a,q)\right]da$$

Integrating by parts (and suppressing the dependence of g_1,g_2 on q) gives

$$\begin{aligned}
\frac{d\log}{dq}\left\{\frac{\int h(q,a)^\zeta f_1(a)\,da}{\int h(q,a)^\zeta f_2(a)\,da}\right\} &= \zeta\frac{d\log h(q,a)}{dq}\int_{-\infty}^a\left[g_1(u)-g_2(u)\right]du\Big|_{-\infty}^{\infty}-\int\zeta\frac{d^2\log h(q,a)}{dadq}\int_{-\infty}^a\left[g_1(u)-g_2(u)\right]du \\
&= \int\zeta\frac{d^2\log h(q,a)}{dadq}\int_{-\infty}^a\left[g_2(u)-g_1(u)\right]duda
\end{aligned}$$

Since h is log supermodular, $\frac{d^2 \log h(q,a)}{dadq} > 0$. We next show that for each a, $\int_{-\infty}^{a} [g_2(u) - g_1(u)]\, du < 0$. Note that

$$\int_{-\infty}^{a} [g_2(u) - g_1(u)]\, du = \int_{-\infty}^{a} \left[\frac{g_2(u)}{g_1(u)} - 1\right] g_1(u)\, du$$

Note that $\int_{-\infty}^{\infty} [g_2(u) - g_1(u)]\, du = 0$. Second, note that $\frac{g_2(u)}{g_1(u)}$ is increasing because $\frac{f_2(u)}{f_1(u)}$ is increasing. Thus there must be some a^* such that $\frac{g_2(u)}{g_1(u)} < 1$ for $a < a^*$ and $\frac{g_2(u)}{g_1(u)} > 1$ for $a > a^*$. If $a < a^*$,

$$\int_{-\infty}^{a} [g_2(u) - g_1(u)]\, du < 0$$

If $a > a^*$, then

$$\begin{aligned}\int_{-\infty}^{a} [g_2(u) - g_1(u)]\, du &= \int_{-\infty}^{\infty} [g_2(u) - g_1(u)]\, du - \int_{a}^{\infty} [g_2(u) - g_1(u)]\, du \\ &= 0 - \int_{a}^{\infty} [g_2(u) - g_1(u)]\, du < 0\end{aligned}$$

■

Lemma E.4 *Among plants of type a and productivity x that have never been grafted, the fraction that are owned by a firm with q is*

$$\frac{h(q,a)^{\zeta}\, n(q)\, F'(q)}{\int h(\tilde{q},a)^{\zeta}\, n(\tilde{q})\, F'(\tilde{q})\, d\tilde{q}}$$

Among plants of type a that have been grafted at least once and that have productivity x, the fraction that are owned by a firm with q is

$$\frac{h(q,a)^{\zeta}\, s(q)\, F'(q)}{\int h(\tilde{q},a)^{\zeta}\, s(\tilde{q})\, F'(\tilde{q})\, d\tilde{q}}$$

Both of these distributions log supermodular in in q and a.

Proof. The measure of denovo plants of type a with initial productivity x created by firms with q (per firm) is $h(q,a)^{\zeta}\, n(q)\, \Upsilon F'(q)\, \zeta x^{-\zeta-1}$. Thus the fraction of plants of type a and productivity x is

$$\frac{h(q,a)^{\zeta}\, n(q)\, F'(q)}{\int h(\tilde{q},a)^{\zeta}\, n(\tilde{q})\, F'(\tilde{q})\, d\tilde{q}}$$

Note that this is the same for each x. Finally, note that the evolution of x and the arrival of an offloading event depend only on x and a, not on the identity or type of firm.

Similarly ,the measure of newly grafted plants of type a with initial productivity x grafted by firms of type q is (per firm)

$$h(q,a)^{\zeta}\, s(q)\, \frac{M(J,S)}{S} F'(q)\, \zeta x^{-\zeta-1} G_a(u)$$

Thus the fraction of plants of type a and productivity x is

$$\frac{h\left(q,a\right)^{\zeta}s\left(q\right)F'\left(q\right)}{\int h\left(\tilde{q},a\right)^{\zeta}s\left(\tilde{q}\right)F'\left(\tilde{q}\right)d\tilde{q}}$$

and similarly, the evolution of x and the arrival of an offloading event depend only on x and a, not on the identity or type of firm.

Lastly, log supermodularity follows from the log supermodularity of h. ■

Lemma E.5 *If b_a is increasing in a, then the fraction of plants that have been grafted is increasing a.*

Proof. For any plant, let $u = \frac{x}{\underline{x}_a}$ be the productivity relative to the exit threshold. u follows a geometric brownian motion with upward jumps at each grafting event. The initial distribution of u among denovo plants is $\zeta u^{-\zeta-1}$ for $u > 1$ for every plant type a. Conditional on not being grafted, u follows the same geometric brownian motion. The arrival rate of a grafting event depends only on u and b_a (which governs the arrival of a potential grafter that would offer u'). The larger b_a, the higher the arrival of a grating event along a sample path of u. Thus higher b_a raises the outflow from nevergrafted status. Further, the fraction of firms that survive past any age is increasing with higher b_a, as the arrival of potential grafters can only raise productivity. Together, this means when b_a, at any age, the fraction of a cohort that has never been grafted is smaller while the fraction that has been grafted is higher, and the fraction that has exited is smaller. ■

Lemma E.6 *For a firm of type q, the density across plant types of the flow of denovo plants is*

$$\frac{h\left(q,a\right)^{\zeta}\underline{x}_a^{-\zeta}F'_{a,0}\left(a\right)}{\int h\left(q,\tilde{a}\right)^{\zeta}\underline{x}_{\tilde{a}}^{-\zeta}dF_{a,0}\left(\tilde{a}\right)}$$

and the density across plant types of the flow of grafted plants is

$$\frac{h\left(q,a\right)^{\zeta}\underline{x}_a^{-\zeta}\int_1^{\infty}\zeta u^{-\zeta-1}G_a\left(\underline{x}_a u\right)du}{\int h\left(q,\tilde{a}\right)^{\zeta}\underline{x}_{\tilde{a}}^{-\zeta}\int_1^{\infty}\zeta u^{-\zeta-1}G_{\tilde{a}}\left(\underline{x}_a u\right)dud\tilde{a}}$$

The ratio of the two is log modular.

Proof. The measure of denovo plants of type a with initial productivity x created by firms with q (per firm) is $h\left(q,a\right)^{\zeta}n\left(q\right)\Upsilon F'\left(q\right)\zeta x^{-\zeta-1}$. Thus the fraction of de novo plants that of type a among firms of type q is

$$\frac{\int_{\underline{x}_a}^{\infty}h\left(q,a\right)^{\zeta}n\left(q\right)\Upsilon F'\left(q\right)\zeta x^{-\zeta-1}dxF'_{a,0}\left(a\right)}{\int\int_{\underline{x}_{\tilde{a}}}^{\infty}h\left(q,\tilde{a}\right)^{\zeta}n\left(q\right)\Upsilon F'\left(q\right)\zeta x^{-\zeta-1}dxdF_{a,0}\left(\tilde{a}\right)} = \frac{h\left(q,a\right)^{\zeta}\underline{x}_a^{-\zeta}F'_{a,0}\left(a\right)}{\int h\left(q,\tilde{a}\right)^{\zeta}\underline{x}_{\tilde{a}}^{-\zeta}dF_{a,0}\left(\tilde{a}\right)}$$

The measure of grafted plants of type a with initial productivity x grafted by firms with q (per firm) is $h\left(q,a\right)^{\zeta}n\left(q\right)\Upsilon F'\left(q\right)\zeta x^{-\zeta-1}G_a\left(x\right)$. Thus the fraction of de novo plants that of type a among firms of type

q is

$$\frac{\int_{\underline{x}_a}^{\infty} h(q,a)^{\zeta} s(q) MF'(q) \zeta x^{-\zeta-1} G_a(x) dx}{\int \int_{\underline{x}_{\tilde{a}}}^{\infty} h(q,\tilde{a})^{\zeta} s(q) MF'(q) \zeta x^{-\zeta-1} G_{\tilde{a}}(x) dx d\tilde{a}} = \frac{h(q,a)^{\zeta} \int_{\underline{x}_a}^{\infty} \zeta x^{-\zeta-1} G_a(x) dx}{\int h(q,\tilde{a})^{\zeta} \int_{\underline{x}_{\tilde{a}}}^{\infty} \zeta x^{-\zeta-1} G_{\tilde{a}}(x) dx d\tilde{a}}$$

$$= \frac{h(q,a)^{\zeta} \underline{x}_a^{-\zeta} \int_1^{\infty} \zeta u^{-\zeta-1} G_a(\underline{x}_a u) du}{\int h(q,\tilde{a})^{\zeta} \underline{x}_{\tilde{a}}^{-\zeta} \int_1^{\infty} \zeta u^{-\zeta-1} G_{\tilde{a}}(\underline{x}_a u) du d\tilde{a}}$$

■

Lemma E.7 *Upon entering a firm, the ratio of the density of productivity among grafted plants to the density of productivity among de novo plants is increasing in productivity. Each of these ratios is independent of firm type.*

Proof. The flow of denovo plants with productivity x and type a for a firm of type q is

$$n(q) \Upsilon h(q,a)^{\zeta} \zeta x^{-\zeta-1} F'_{a,0}(a)$$

so that the density is $\zeta \underline{x}_a^{\zeta} x^{-\zeta-1}$ for $x > \underline{x}_a$. The flow of grafted plants with productivity x and type a for a firm of type q is

$$s(q) h(q,a)^{\zeta} \zeta x^{-\zeta-1} G_a(x)$$

so that the density is

$$\frac{\zeta x^{-\zeta-1} G_a(x)}{\int_{\underline{x}_a}^{\infty} \zeta u^{-\zeta-1} G_a(u) du}$$

The ratio is

$$\frac{G_a(x)}{\underline{x}_a^{\zeta} \int_{\underline{x}_a}^{\infty} \zeta u^{-\zeta-1} G_a(u) du}$$

which is increasing in x. ■

Proposition E.8 *The distribution of productivity and plant sizes (by any measure) among newly grafted plants (in their new firms) stochastically dominates the distributions of de novo plants*

Proof. Probability larger than x conditional on a: $(1-\gamma_a) \underline{x}_a^{\zeta} x^{-\zeta} + \gamma_a \frac{\zeta x^{-\zeta-1} G_a(x)}{\int_{\underline{x}_a}^{\infty} \zeta u^{-\zeta-1} G_a(u) du}$

$$\Pr(X > x | a, \text{De novo})$$

$$\Pr(X > x | a, \text{grafted})$$

$$\begin{aligned}
\Pr(X > x|\text{De novo}) &= \int \Pr(X > x|a, \text{De novo}) \Pr(a|\text{ De novo})\, da \\
&\leq \int \Pr(X > x|a, \text{De novo}) \Pr(a|\text{ grafted})\, da \\
&\leq \int \Pr(X > x|a, \text{grafted}) \Pr(a|\text{ grafted})\, da \\
&= \Pr(X > x|\text{grafted})
\end{aligned}$$

where the first and last lines use the law of iterated probabilities, the second line uses the facts that (i) $\Pr(X > x|a, \text{De novo}) = \underline{x}_a^{\zeta} x^{-\zeta}$ for $x \geq \underline{x}_a$ is increasing in a, and (ii) $\Pr(a|\text{ grafted})$ stochastically dominates $\Pr(a|\text{ De novo})$ because the fraction of plants that have been grafted is increasing in a. ■

Proposition E.9 *De novo plants of higher q firms are larger at birth and tend to be higher a.*

Proof. The measure of denovo plants of type a with initial productivity x created by firms with q (per firm) is $h(q,a)^{\zeta} n(q) \Upsilon F'(q) \zeta x^{-\zeta-1}$. For a firm of type a and productivity x, labor for production is $\bar{l}x$, with $\bar{l} = \frac{1-\alpha}{\alpha} w^{-\frac{1}{\alpha}}$. Then average size of these plants is

$$\frac{\int \int_{\underline{x}_a}^{\infty} (\bar{l}x)\, h(q,a)^{\zeta} n(q) \Upsilon F'(q) \zeta x^{-\zeta-1} dx dF_a(a)}{\int_a \int_{\underline{x}_a}^{\infty} h(q,a)^{\zeta} n(q) \Upsilon F'(q) \zeta x^{-\zeta-1} dx dF_a(a)} = \bar{l} \frac{\int h(q,a) \int_{\underline{x}_a}^{\infty} (x) \zeta x^{-\zeta-1} dx dF(a)}{\int h(q,a) \int_{\underline{x}_a}^{\infty} \zeta x^{-\zeta-1} dx dF(a)} = \bar{l} \frac{\int h(q,a)\, \underline{x}_a^{1-\zeta} \frac{\zeta}{\zeta-1} dF_a(a)}{\int h(q,a)\, \underline{x}_a^{-\zeta} dF_a(a)} = \bar{l}$$

The result follows from the log supermodularity of h and the fact that $\underline{x}_a$ is increasing. ■

Proposition E.10 *Let $\psi(q,a)$ be the measure of plants of type a operated by firms of type q. ψ is log supermodular.*

Among active plants of type a, let γ_a be the fraction of that have been grafted in the past. The fraction of plants of type a that are operated by firms of type q is

$$(1-\gamma_a) f_1(q,a) + \gamma_a f_2(q,a)$$

$$\frac{d}{da}\left(\frac{(1-\gamma_a)\frac{\partial f_1(q,a)}{\partial q} + \gamma_a \frac{\partial f_2(q,a)}{\partial q}}{(1-\gamma_a) f_1(q,a) + \gamma_a f_2(q,a)}\right) = \frac{(1-\gamma_a)\frac{\partial f_1(q,a)}{\partial q} + \gamma_a \frac{\partial f_2(q,a)}{\partial q}}{(1-\gamma_a) f_1(q,a) + \gamma_a f_2(q,a)} - \frac{\frac{(1-\gamma_a) f_1(q,a) + \gamma_a f_2(q,a)}{} \left[(1-\gamma_a)\frac{\partial f_1(q,a)}{\partial q} + \gamma_a \frac{\partial f_2(q,a)}{\partial q}\right.}{\left[(1-\gamma_a) f_1(q,a) + \gamma_a f_2(q,a)\right]^2}$$

$$(1-\gamma_a) \frac{h(q,a)^{\zeta} n(q) F'(q)}{\int h(\tilde{q},a)^{\zeta} n(\tilde{q}) F'(\tilde{q})\, d\tilde{q}} + \gamma_a \frac{h(q,a)^{\zeta} s(q) F'(q)}{\int h(\tilde{q},a)^{\zeta} s(\tilde{q}) F'(\tilde{q})\, d\tilde{q}}$$

Proposition E.11 *Higher a plants are more likely to survive*

Proposition E.12 *Plant size increases after grafting event*

E.2 Tail Behavior

F Planner's Problem

Let $\tilde{g}(a,x) \equiv \tilde{G}'_a(x)$ be the density of plants of type a and effective productivity x. Let $\mathcal{W}$ be a planner's objective function, defined as

$$\rho\mathcal{W}\left(\{\tilde{g}(a,x)\},J,L\right) = \sup_{n(\cdot),s(\cdot),C,\underline{x}_a,l_a(\cdot)} Lu\left(\frac{C}{L}\right) + \int\int \mathcal{W}_{a,x}\frac{\partial\tilde{g}(a,x)}{\partial t}dadx + \mathcal{W}_J\frac{\partial J}{\partial t} + \mathcal{W}_L\dot{L},$$

where we use the shorthand $\mathcal{W}_{a,x}$ to indicate the directional derivative of $\mathcal{W}$ in the direction of $\tilde{g}(a,x)$. The planner faces the resource constraints:

$$\xi_C : C \leq \int\int \frac{1}{\alpha^\alpha\left(1-\alpha\right)^{1-\alpha}}x^\alpha l_a\left(x\right)^{1-\alpha}\tilde{g}\left(a,x\right)dxda,$$

$$\xi_L : \int\int\left[l_a\left(x\right)+\Phi_a\right]\tilde{g}(a,x)dxda + J\int\left[c\left(n\left(q\right)\right)+c\left(s\left(q\right)\right)\right]dF_q\left(q\right) + \Phi^E\frac{\partial J}{\partial t} \leq L,$$

$$\begin{aligned}
\frac{\partial\tilde{g}(a,x)}{\partial t} &= -\frac{d}{dx}\left[\mu x\tilde{g}(a,x)\right] + \frac{d^2}{dx^2}\left[\frac{\sigma^2}{2}x^2\tilde{g}(a,x)\right] \\
&\quad +\Upsilon\zeta x^{-\zeta-1}J\int h\left(q,a\right)^\zeta n\left(q\right)F'_a\left(a\right)F'_q\left(q\right)dq \\
&\quad -\tilde{g}(a,x)B_a x^{-\zeta} + \zeta x^{-\zeta-1}B_a\int_{\underline{x}_a}^x\tilde{g}\left(a,u\right)du,
\end{aligned}$$

$$\begin{aligned}
\tilde{g}\left(a,\underline{x}_a\right) &= 0, \\
\int_{\underline{x}_a}^\infty\tilde{g}(a,x) &< \infty, \\
\lim_{x\to\infty}\mu x\tilde{g}(a,x) - \frac{d}{dx},\left[\frac{\sigma^2}{2}x^2\tilde{g}(a,x)\right] &= 0,
\end{aligned}$$

and where $B_a\left(\{s\left(q\right)\}\right) \equiv b_a\underline{x}_a^\zeta$ is defined as

$$B_a\left(\{s\left(q\right)\}\right) \equiv M\left(1,\theta\right)\overline{h\left(\cdot,a\right)^\zeta},$$

with

$$\theta \equiv \int s(q)\, dF_q(q),$$
$$\overline{h(\cdot,a)^{\zeta}} \equiv \frac{\int h(q,a)^{\zeta} s(q)\, dF_q(q)}{\int s(q)\, dF_q(q)}.$$

Note that the derivatives of B_a are

$$\begin{aligned} \frac{dB_a}{ds(q)} &= M_S(1,\theta)\,\overline{h(\cdot,a)^{\zeta}} F_q'(q) + M(1,\theta)\left(\frac{h(q,a)^{\zeta}}{\theta} - \frac{\overline{h(\cdot,a)^{\zeta}}}{\theta}\right) F_q'(q) \\ &= \left\{ h(q,a)^{\zeta} - \left(1 - \frac{\theta M_S(1,\theta)}{M(1,\theta)}\right)\overline{h(\cdot,a)^{\zeta}}\right\}\frac{M(1,\theta)}{\theta} F_q'(q). \end{aligned}$$

$\mathcal{W}$ is homogeneous of degree one, so we can write the last term of the objective function as

$$\mathcal{W}_L \dot{L} = \mathcal{W}_L L g_L = \left(\mathcal{W} - \mathcal{W}_J J - \int\int \mathcal{W}_{a,x}\tilde{g}(a,x)\,da\,dx\right) g_L.$$

The planner's problem can thus be expressed as

$$(\rho - g_L)\,\mathcal{W}(\{\tilde{g}(a,x)\}, J, L) = \sup_{n(\cdot),s(\cdot),C,\underline{x}_a,l_a(\cdot)} Lu\left(\frac{C}{L}\right) + \int\int \mathcal{W}_{a,x}\left(\frac{\partial \tilde{g}(a,x)}{\partial t} - g_L\tilde{g}(a,x)\right) da\,dx + \mathcal{W}_J\left(\frac{\partial J}{\partial t} - g_L J\right).$$

The first order conditions are

$$\begin{aligned} C &: u'\left(\frac{C}{L}\right) = \xi_C \\ l_a(x) &: \xi_C \frac{1}{\alpha^{\alpha}(1-\alpha)^{-\alpha}} x^{\alpha} l_a(x)^{-\alpha} = \xi_L \\ n(q) &: \xi_L J c'(n(q)) = J\int\int \mathcal{W}_{a,x}\Upsilon\zeta x^{-\zeta-1} h(q,a)^{\zeta} F_a'(a)\,da\,dx \\ s(q) &: \xi_L J c'(s(q)) F_q'(q) = \int \frac{dB_a}{ds(q)}\int \mathcal{W}_{a,x}\left[-\tilde{g}(a,x)x^{-\zeta} + \zeta x^{-\zeta-1}\int_{\underline{x}_a}^{x}\tilde{g}(a,u)\,du\right] dx\,da \end{aligned}$$

Define $v_a^*(x) = \frac{\mathcal{W}_{a,x}}{\xi_L}$ and $w^* = \frac{\xi_L}{\xi_C}$. The envelope condition for J (after imposing steady state of $\frac{1}{J}\frac{dJ}{dt} = g_L$)

gives

$$(\rho - g_L)\,\mathcal{W}_J = -\mathcal{W}_J g_L - \xi_L\int [c(n(q)) + c(s(q))]\, dF_q(q) + \int\int\int \mathcal{W}_{a,x}\Upsilon\zeta x^{-\zeta-1} h(q,a)^{\zeta} n(q)\, dF_q(q)\,dx\,dF_a(a).$$

Simplifying gives

$$\rho\mathcal{W}_J = -\xi_L \int \left[c\left(n\left(q\right)\right) + c\left(s\left(q\right)\right)\right] dF_q\left(q\right) + \int\int\int \mathcal{W}_{a,x}\Upsilon\zeta x^{-\zeta-1} h\left(q,a\right)^{\zeta} n\left(q\right) dF_q(q)dxdF_a(a).$$

Using the FOC for $\frac{\partial J}{\partial t}$, this is

$$\rho\Phi^E = \int \left\{-c\left(n(q)\right) - c\left(s(q)\right) + \Upsilon n\left(q\right) \int\int v_a^*\left(x\right)\zeta x^{-\zeta-1} h\left(q,a\right)^{\zeta} dxdF_a(a)\right\} dF_q(q).$$

The envelope condition for $\tilde{g}(a,x)$ (after imposing steady state of $\frac{1}{\tilde{g}(a,x)}\frac{\partial\tilde{g}(a,x)}{dt} = g_L$) for any interior $x \in (\underline{x}_a, \infty)$ is

$$\begin{aligned}
\left(\rho - g_L\right)\mathcal{W}_{a,x} &= -g_L\mathcal{W}_{a,x} + \xi_C \frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}} x^{\alpha} l_a\left(x\right)^{1-\alpha} - \xi_L\left[l_a\left(x\right) + \Phi_a\right] \\
&\quad + \int\int \mathcal{W}_{a,u}\left[-\frac{d}{du}\left[\mu u\delta_x\left(u\right)\right] + \frac{d^2}{du^2}\left[\frac{\sigma^2}{2}u^2\delta_x\left(u\right)\right]\right] duda \\
&\quad - \mathcal{W}_{a,x}B_a x^{-\zeta} + \int_x^{\infty} \mathcal{W}_{a,u}\zeta u^{-\zeta-1}B_a du,
\end{aligned}$$

where $\delta_x\left(\cdot\right)$ is the dirac delta function. We can rearrange the second line by integrating by parts, once for the first term and twice for the second term

$$\int_{\underline{x}_a}^{\infty} \mathcal{W}_{a,u}\left[-\frac{d}{du}\left[\mu u\delta_x\left(u\right)\right] + \frac{d^2}{du^2}\left[\frac{\sigma^2}{2}u^2\delta_x\left(u\right)\right]\right] du = \frac{\partial\mathcal{W}_{a,x}}{\partial x}\mu x + \frac{\partial^2\mathcal{W}_{a,x}}{\partial x^2}\frac{\sigma^2}{2}x^2.$$

Together, the envelope condition becomes

$$\begin{aligned}
\rho\mathcal{W}_{a,x} &= \xi_C \frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}} x^{\alpha} l_a\left(x\right)^{1-\alpha} - \xi_L\left[l_a\left(x\right) + \Phi_a\right] \\
&\quad + \frac{\partial\mathcal{W}_{a,x}}{\partial x}\mu x + \frac{\partial^2\mathcal{W}_{a,x}}{\partial x^2}\frac{\sigma^2}{2}x^2 \\
&\quad + \int_x^{\infty}\left[\mathcal{W}_{a,u} - \mathcal{W}_{a,x}\right]\zeta u^{-\zeta-1}B_a du.
\end{aligned}$$

The FOC for $l_a\left(x\right)$ gives $l_a\left(x\right) = \frac{1-\alpha}{\alpha}\left(w^*\right)^{-\frac{1}{\alpha}} x$, so that

$$\frac{\xi_C}{\xi_L}\frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}} x^{\alpha} l_a\left(x\right)^{1-\alpha} - l_a\left(x\right) = \left(w^*\right)^{-\frac{1}{\alpha}} x.$$

The HJB for $v_a^*(x)$

$$\rho v_a^*(x) = (w^*)^{-\frac{1}{\alpha}} x - \Phi_a + v_a^{*\prime}(x)\,\mu x + v_a^{*\prime\prime}(x)\,\frac{\sigma^2}{2} x^2 + B_a \int_x^\infty \left[v_a^*(u) - v_a^*(x)\right] \zeta u^{-\zeta-1} du.$$

The FOCs for $n(q)$ and $s(q)$ can be expressed, after dividing by $J\xi_L F_q'(q)$ and using $g(a,x) \equiv \frac{\tilde{g}(a,x)}{J}$ as

$$\begin{aligned} n(q) \quad &: \quad c'(n(q)) = \int\int v_a^*(x)\,\Upsilon\zeta x^{-\zeta-1} h(q,a)^\zeta F_a'(a)\,dadx \\ s(q) \quad &: \quad c'(s(q)) = \int \frac{1}{F_q'(q)} \frac{dB_a}{ds(q)} \int_{\underline{x}_a}^\infty v_a^*(x) \left[-g(a,x)\,x^{-\zeta} + \zeta x^{-\zeta-1} \int_{\underline{x}_a}^x g(a,u)\,du\right] dxda \end{aligned}$$

For the second equation, the first half of the inner integral can be rearranged by changing the order of integration then swapping the roles of the variables gives

$$\begin{aligned} \int_{\underline{x}_a}^\infty v_a^*(x)\left[-g(a,x)x^{-\zeta}\right] dx &= -\int_{\underline{x}_a}^\infty v_a^*(x)\,g(a,x) \int_x^\infty \zeta u^{-\zeta-1} dudx \\ &= -\int_{\underline{x}_a}^\infty \int_{\underline{x}_a}^u v_a^*(x)\,g(a,x)dx\zeta u^{-\zeta-1} du \\ &= -\int_{\underline{x}_a}^\infty \int_{\underline{x}_a}^x v_a^*(u)\,g(a,u)\,du\zeta x^{-\zeta-1} dx. \end{aligned}$$

So that the inner integral can be expressed as

$$\int_{\underline{x}_a}^\infty v_a^*(x)\left[-g(a,x)x^{-\zeta} + \zeta x^{-\zeta-1}\int_{\underline{x}_a}^x g(a,u)\,du\right] dx = \int_{\underline{x}_a}^\infty \int_{\underline{x}_a}^x \left[v_a^*(x) - v_a^*(u)\right] g(a,u)\,du\zeta x^{-\zeta-1} dx.$$

Using this along with the expression for $\frac{dB_a}{ds(q)}$ gives that the FOC wrt $s(q)$ is

$$c'(s(q)) = \frac{M(1,\theta)}{\theta} \int \left\{ h(q,a)^\zeta - \left(1 - \frac{\theta M_S(1,\theta)}{M(1,\theta)}\right) \overline{h(\cdot,a)^\zeta} \right\} \int_{\underline{x}_a}^\infty \int_{\underline{x}_a}^x \left[v_a^*(x) - v_a^*(u)\right] g(a,u)\,du\zeta x^{-\zeta-1} dxda.$$

G Log Modularity Taxes and Transfers

We construct a set of firm-plant-specific transfers that incentivize firms to behave as if their productivity is log-modular rather than log-supermodular, while ensuring that the actual production function retains the assumed log-supermodularity. Consider the system of taxes and subsidies defined by

$$\mathcal{T}(q,a) \equiv \left[\frac{\tau_f(q)\tau_p(a)}{h(q,a)}\right]^\alpha,$$

where the functions $\tau_p(a)$ and $\tau_f(q)$ are chosen so that the tax scheme is revenue-neutral—or output-neutral given that firms are price-takers in the goods market—for both plant and firm types. Under these distortions, a type a plant belonging to a type q firm with match productivity z solves the optimization problem given by

$$\begin{aligned}\pi(q,a,z) &= \max_{\ell\geq 0}\ \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\mathcal{T}(q,a)[h(q,a)z]^\alpha \ell^{1-\alpha} - w\ell - w\Phi(a)\ ,\\ &= \max_{\ell\geq 0}\ \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}[\tau_f(q)\tau_p(a)z]^\alpha \ell^{1-\alpha} - w\ell - w\Phi(a)\ .\end{aligned}$$

A key observation is that distortions transform the optimization problem into one in which output is log-modular in (q,a) rather than log-supermodular. Let $\ell(q,a,z)$ denote the optimal plant's labor choice.

We now characterize $\tau_f(q)$ and $\tau_p(a)$. For the tax scheme to be revenue-neutral for each firm type q we need $\mathcal{T}(q,a)$ to satisfy

$$\iint \left[\mathcal{T}(q,a)-1\right]\left[h(q,a)z\right]^\alpha \ell(q,a,z)^{1-\alpha} dG'_{q,a}(z)dzda = 0\ . \tag{45}$$

We also require the set of distortions $\mathcal{T}(q,a)$ to be revenue-neutral with respect to plant types. This leads to a similar restriction as in (45), but in this case that aggregates across firm types rather than across plant types. The following proposition characterizes the revenue-neutral taxes.

Proposition G.1 *The set of distortions $\mathcal{T}(q,a)$ is revenue-neutral with respect to firm and plant types if and only if*

$$\tau_f(q) = \left[\iint \left(\frac{h(q,a)}{\tau_p(a)}\right)^\alpha \omega_q(\tilde{x},a)d\tilde{x}da\right]^{1/\alpha},\ \text{and}\ \tau_p(a) = \left[\iint \left(\frac{h(q,a)}{\tau_f(q)}\right)^\alpha \omega_a(\tilde{x},q)d\tilde{x}dq\right]^{1/\alpha}\ ,$$

where $\omega_q(\tilde{x},a) \equiv \frac{\tilde{x}g_q(\tilde{x},a)}{\iint \tilde{x}g_q(\tilde{x},a)d\tilde{x}da} \in [0,1]$ *for all* $\tilde{x}$ *and* a *with* $\iint \omega_q(\tilde{x},a)d\tilde{x}da = 1$, *and* $\omega_a(\tilde{x},q) \equiv \frac{\tilde{x}g_q(\tilde{x},a)}{\iint \tilde{x}g_q(\tilde{x},a)d\tilde{x}dq} \in [0,1]$ *for all* $\tilde{x}$ *and* q *with* $\iint \omega_a(\tilde{x},q)d\tilde{x}dq = 1$.

H Computation